ALGEBRA

A TEXTBOOK FOR SECONDARY SCHOOLS

Part I

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MOHAN LAL

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FOREWORD

The man in the street talks today, and rightly, about the Scientific Revolution that shapes our lives. To the educator, the scientific revolution presupposes a revolution in concepts, understandings and teaching methods. Triggering off the scientific revolution came some years ago, an educational revolution in science and mathematics teaching. Gone now are the days when we could teach mathematics as if it were a study in techniques and skills in which the student was content to memorise skills without understanding concepts, and in which the teacher could be satisfied with the transmission of these skills. Today, it is essential to get the child to understand the fundamental concepts of modern mathematics and science and to teach skills for what skills are worth.

We in India have arrived rather late at the world's scientific-cum-educational party. Only recently have we been able to develop an integrated programme for the teaching of mathematics and science, to run Summer Institutes to train teachers in modern mathematics and to push forward the revolution in education that is integral to the business of our scientific revolution. Curriculum and teaching techniques in mathematics at School level have to be integrated with modern mathematics at University level.

In recognition of this need, the National Council of Educational Research and Training has undertaken to produce modern textbooks in mathematics and science. A detailed programme of publication has been undertaken by the Central Committee on Educational Literature that was set up by the National Council in 1961. In addition, the Council sometimes sponsors valuable educational literature, such as the present book that has been written by Shri Shanti Narayan, Principal, Hans Raj College, Delhi. Working at high pressure, Shri Shanti Narayan has produced a modern textbook on Algebra for Secondary Schools. We publish here Part I of the textbook by him and his co-author Shri Mohan Lal, Lecturer at P.G. D.A.V. College, New Delhi.

The aim of Part I of this book in mathematics is to enable the student at secondary school to form a clear understanding of concepts and to develop the skills to solve mathematical problems. Throughout, the presentation of material has been influenced by developments in modern methods. We, in the National Council, hope this book will stimulate and equip students of mathematics all over India to think further for themselves in this vital field.

The National Council places on record its gratitude to Shri Shanti Narayan and to Shri Mohan Lal for the industry and speed with which they have produced this book, which is the result of mature experience and great determination. We look forward with hundreds of other readers, from whom demands have already been received, to Part II. From these readers, Shri Shanti Narayan and the National Council invite suggestions on this book. We would be grateful for constructive and precise suggestions that will enable us to bring out a revised edition when this becomes necessary.

New Delhi

L.S. CHANDRAKANT

PREFACE

There are people all over the world who have been busy creating mathematics consisting of new concepts and techniques. These concepts and techniques have a profound influence on the society which is undergoing revolutionary changes. This growth, however, has had no impact on the Mathematics Instructional Programmes specially at the school level in our country. What is being taught to the students in secondary schools today is exactly the same as was perhaps taught to their great-grandfathers. This means that our instruction has not kept pace with the growth of mathematics. The result is that we have not been able to make as extensive a use of mathematics as others to promote our national development. At the same time we have not been able to participate to the extent possible in the process of creating mathematics.

Modern mathematical thought has begun to be reflected in some post-graduate courses in India. Unfortunately, this is only being super-imposed, and implies the failure to recognise the evolutionary character of mathematical growth. Were this not so, one would expect the school mathematics programmes to be influenced by modern mathematical thought, that is a development of the so-called elementary mathematics. Thus, there has to be a constant inter-communication between the growth of mathematics and mathematical instruction at different levels of schooling. Dissemination of mathematical knowledge with its present rate of doubling every ten years has to be accompanied by a corresponding concern for continuous efforts to improve programmes in mathematical instruction.

Mathematics as presented here will be more stimulating and interesting than the Mathematics that we have been used to teaching in our schools so far, and which has come to be identified only with the learning of some rules. It is important to avoid the feeling, which is unfortunately there, that this presentation will amount to making things more difficult. We should not give way to this feeling and permit it to take hold of us without any experience of the new programme. It may just be a fear of the unknown.

Though an axiomatic presentation of the subject starting with undefined concepts and unproved propositions is not to be the aim at this stage, we hope that the presentation will help students in terms of their present maturity to acquire an understanding of the nature of definitions and the principles of proof making. It is a pity

that the general impression has been that whereas in geometry we prove, in algebra we merely do. This impression is far from true.

To enable the student really to imbibe the spirit of this presentation, he is helped to read the book and not just do the exercises. There is no alternative to a thoughtful reading of the book. The teacher should encourage the student to read and try to involve him in what he has read.

The authors do not suggest that this book, which of course marks a drastic change in presentation and content, is intended to be the last word on what is needed. A trial is, however, earnestly requested. To the extent that this presentation raises fruitful controversy leading to constructive suggestions, it will be successful and worthwhile.

The authors are deeply grateful to the National Council of Educational Research and Training which agreed to publish the book. The hearty co-operation which is forthcoming from all quarters augurs well for improvement. The Council made available the services of Shri R.C. Sharma, Senior Research Officer in the Department of Science Education, who went through the manuscript and made very helpful and useful suggestions. We are grateful to Kumari Nilima for her help in the preparation of this book.

SHANTI NARAYAN

Delhi

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CONTENTS

List of Symbols

Principal Sets

N	The set of natural numbers
F	The set of fractions
I	The set of integers
$\mathbf{Q_0}$	The set of non-zero rationals
Q	The set of rational numbers

Sets and Logic

$\cup$	Union of sets
$\cap$	Intersection of sets
$\in$	Belongs to
$\notin$	Does not belong to
$\subset$	Is a sub-set of
ϕ	Null set
$\forall$	For all
:	Such that
$\Rightarrow$	Implies
$\Leftarrow$	Is implied by
$\Leftrightarrow$	Is equivalent to
$\exists$	There exists

Composition

$+$	Addition
$\times$	Multiplication
$-$	Subtraction
$\div$	Division

Relations

$=$	Is equal to
$\neq$	Is not equal to
$>$	Is greater than
$<$	Is less than
$\geqslant$	Is greater than or equal to
$\leqslant$	Is less than or equal to
$\ngtr$	Is not greater than
$\nless$	Is not less than
$\mid$	Is a factor of
$\nmid$	Is not a factor of

1

Natural Numbers
Compositions and Relations

1. INTRODUCTION

A child has his first contact with Mathematics as soon as his schooling starts through what are called

Counting Numbers or *Natural Numbers.*

The child develops his awareness of counting numbers through association with different collections of objects. He learns to associate with each collection of a single object, the number *one*. Again on adjoining a new object to a collection containing a single object, there is obtained another collection to which is associated, the number *two*. Further as we go on adjoining new objects one by one to the collections already obtained, we have collections to which are associated successively the numbers

three, four, five, six, seven ..

The collection or what is now usually called the *set* of numbers thus obtained is called *the set of natural numbers.*

The process of adjoining new objects is obviously endless. This means that there is no last natural number and that there is a natural number succeeding any given natural number. This idea is expressed by saying that the set of natural numbers is *infinite*.

We thus need an endless number of symbols, one for each natural number, *to denote the members* of the infinite set of natural numbers. Hence we find it necessary to have a technique for representing the natural numbers by means of a finite number of basic symbols. Moreover, in the interest of intercommunication at different levels, this technique must be scientifically conceived and command international acceptance. It is obvious that the employment of different techniques by different individuals will lead to chaos.

Luckily for the world civilization as such, a scheme which meets both these demands was conceived by the Hindus in India. This scheme, known as the *Positional Scheme* enables us to express any given natural number in terms of any finite number of symbols. Moreover, its adoption renders very convenient the processes of addition and multiplication of numbers.

The idea underlying the positional scheme makes it possible to represent every natural number in terms of two or more basic symbols. Depending upon different numbers of basic symbols employed, we have different schemes. These schemes are also known as different *Systems of Numeration* The usual scheme, called the *Decimal Scheme* employs the nine symbols

$$1, 2, 3, 4, 5, 6, 7, 8, 9$$

together with the symbol '0' which the Hindus called *Shunya* (शून्य) and which, in the English language, is called 'Zero'. As an example, we see that the number *three hundred and forty-seven* in the decimal scheme is written as

$$347$$

so that we have
$$347 = 3 \times 10 \times 10 + 4 \times 10 + 7$$
$$= 3 \times 100 + 4 \times 10 + 7$$

and the symbols 3, 4, 7 respectively, stand for

3 hundreds, 4 tens, and 7 ones.

Interchanging the positions of the symbols 3, 4, 7, we have

$$374 = 3 \times 10 \times 10 + 7 \times 10 + 4 = 3 \times 100 + 7 \times 10 + 4$$
$$437 = 4 \times 10 \times 10 + 3 \times 10 + 7 = 4 \times 100 + 3 \times 10 + 7$$
$$473 = 4 \times 10 \times 10 + 7 \times 10 + 3 = 4 \times 100 + 7 \times 10 + 3$$
$$734 = 7 \times 10 \times 10 + 3 \times 10 + 4 = 7 \times 100 + 3 \times 10 + 4$$
$$743 = 7 \times 10 \times 10 + 4 \times 10 + 3 = 7 \times 100 + 4 \times 10 + 3.$$

Of the three digits, the first one on the right denotes so many ones, the next to its left denotes so many tens and the one next to its left so many ten times ten.

As another example we see that the natural number *three hundred and four*, is, in the decimal scheme, denoted as

$$304$$

so that we have

four ones, no ten, and three ten times ten.

In other words

$$304 = 3 \times 10 \times 10 + 4 = 3 \times 100 + 4.$$

The student may similarly write down a few more numbers in the expanded form. Let him, for example, write in the expanded form the numbers denoted by 587, 32, 5329.

Essentially there is nothing sacred about the employment of any *particular number* of basic symbols for the representation of natural numbers in terms of the

positional scheme. We could have any number of them and different number of symbols will give rise to different systems of numeration. While the decimal scheme has been and continues to be employed for almost all purposes, recently the *Binary Scheme* employing only the two symbols

$$0, 1$$

has been found to be of great scientific interest. It is in terms of this *Binary Scheme* that the High Speed Computors function. It may be mentioned that the recent development of these computors, capable of carrying out all sorts of numercial calculations at surprisingly large speed has led to new vistas of scientific resarch and enquiry. For instance, the immensity of calculations involved in the designing of sputniks would have been completely beyound human reach but for the computors. It may also be mentioned that the world has now in its possession the values of π up to 100264 decimal places obtained by a computor in 8 hours and 40 minutes.

Unless otherwise stated we shall be employing the decimal scheme all along the book.

Numbers and Numerals. We sometimes make a distinction between NUMBERS and NUMERALS. While the number is a concept, a numeral is a representation of the same. We shall naturally have several different numerals to represent the same number. For example

$$2 + 4, 2 \times 3, 9 - 3, 24 \div 4, 6$$

are different numerals for the same number 6.

EXERCISES

In each of the following, check whether the numerals represent the same number or not.

(*i*) 8×7 and $20 - 5$ (*ii*) 9×3 and 39
(*iii*) $8 \div 2$ and $6 - 2$ (*iv*) $9 + (3 \times 2)$ and 12
(*v*) $12 - (5 \times 2)$ and $4 \div 2$ (*vi*) $36 \div (4 + 2)$ and $(36 \div 4) + (36 \div 2)$
(*vii*) $9 + (7 - 2)$ and $2 + (36 \div 3)$ (*viii*) $(3 + 2) + 4$ and $3 + (2 + 4)$
(*ix*) $2 \times (9 - 6)$ and $(2 \times 9) - (2 \times 6)$.

It may be pointed out that different positional schemes lead to different numerals as representatives of the same number. This is treated in detail in an appendix at the end of the book.

We also have Greek, Roman and Arabic numerals as different representatives of the same number.

While conceptually the distinction between Number and Numeral is there and is important, we shall, however, in usage, not make this distinction.

2. BASIC LAWS OF ADDITION AND MULTIPLICATION COMPOSITIONS

Assuming that the student is, at this stage, well versed in carrying out any given computations involving addition and multiplication, we shall in this chapter

draw his attention to the *basic properties* underlying the manner of his doing these computations. We will here attempt to have a *New Look* at the various compositions and recognize some fundamental laws in existence. These fundamental laws, like every other fundamental law, are simple looking but with a very profound and far reaching significance. The obviousness and the apparent triviality of these laws must not lead us to undermine their fundamental character. We shall only be *identifying and stating these laws*. It may be remarked that underlying the processes of carrying out the addition and multiplication compositions, in terms of various steps and directions which we learn to do in a mechanical manner, is the existence and operation of these laws. Finally we may state that these laws provide the rationale for the mechanics of the computing processes.

Addition Composition. Let us consider any two collections, say, of 7 books and 4 books. We adjoin to the seven-book collection, the four-book collection to obtain another collection to which we associate the natural number $7 + 4$, called the sum of the two natural numbers 7 and 4, taken in this very order. Thus to the pair of natural numbers 7 and 4, taken in this order, we associate the natural number $7 + 4$, called their sum. Similarly to the pair of natural numbers a and b taken, in this given order, we associate the natural number

$$a + b,$$

called the sum of a and b in this very order.

We say that the *Addition Composition* has been defined in the set of natural numbers inasmuch as, to any two natural numbers taken in a certain order, a, b we can associate a natural number, called their sum denoted symbolically by

$$a + b.$$

Commutative Property of Addition. If, in the above illustration of collections of books, we had adjoined to the four-book collection, the seven-book collection, the resulting collection will have the same number of books as the first one. We describe this position by writing

$$4 + 7 = 7 + 4$$

and notice that interchanging or commuting the position of 4 and 7 does not alter the sum. Thus,

$$4 + 7 = 7 + 4$$

is a *statement* which is obviously true. A statement which is true will be referred to simply as **True Statement.** A statement which is not true will be described as a **False Statement.**

Each of the following

$$2 + 3 = 3 + 2, \qquad 7 + 9 = 9 + 7, \qquad 24 + 37 = 37 + 24$$

is a true starement.

Thus, we recognize the first basic property of addition called the *commutative property, principle* or *law of addition.*

We should note that each of the true statements above is only an illustration of this commutative property and *not* the statement of the property as such which we give below :

COMMUTATIVE PROPERTY OF ADDITION

$$a + b = b + a$$

for all natural numbers a, b.

Here a, b do not stand for any particular natural numbers. The equality will be true whatever natural numbers be substituted for a and b. For example, if

$$a = 3 \text{ and } b = 5$$

we have $\quad 3 + 5 = 5 + 3$

is true.

EXERCISES

1. Name the property of addition on the basis of which the following statements are true.

(*i*) $25 + 27 = 27 + 25$ (*ii*) $37 + 44 = 44 + 37$

(*iii*) $98 + 75 = 75 + 98$ (*iv*) $77 + 9 = 9 + 77$.

2. Give some statements, which are true because of the commutative property of addition.

3. Each of the following is a true statement. What is the natural number x in each case ?

(*i*) $12 + 17 = 17 + x$ (*ii*) $29 + x = 15 + 29$

(*iii*) $x + 44 = 44 + 19$ (*iv*) $77 + 9 = x + 77$

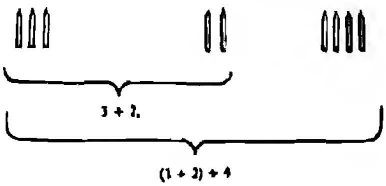
(*v*) $x + 21 = 21 + 39$ (*vi*) $105 + 4 = 4 + x$

(*vii*) $47 + 33 = x + 47$ (*viii*) $25 + x = 14 + 25$.

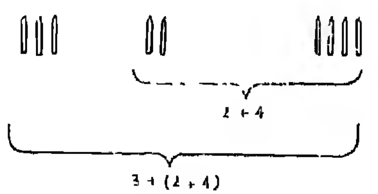
Associative Property of Addition. We shall now recognize and formulate a property of addition which pertains to three natural numbers as against the commutative property pertaining to two natural numbers.

Suppose we have three collections of, say, three, two and four pencils each. We consider the following two different processes.

I. We adjoin the two-pencil collection to the three-pencil collection and adjoin the four-pencil collection to the one thus obtained.



II. We adjoin the four-pencil collection to the two-pencil collection and adjoin the collection thus obtained to the remaining three-pencil collection.



The number of pencils in each of the two collections obtained by the two different processes is the same, so that we have

$$(3 + 2) + 4 = 3 + (2 + 4).$$

It will be seen that while we have not changed the position of any of the three numbers, we have only had different manners of association of the numbers on the two sides. We have here an instance of what is called the *Associative Property of Addition*, which we state as follows :

ASSOCIATIVE PROPERTY OF ADDITION

$$(a + b) + c = a + (b + c)$$

for all natural numbers a, b, c.

As an application of the associative property of addition, we see that each of the following is a true statement.

$$\begin{aligned}
(i)\ & (7 + 5) + 9 &&= 7 + (5 + 9)\\
(ii)\ & (23 + 12) + 8 &&= 23 + (12 + 8)\\
(iii)\ & (11 + 55) + 44 &&= 11 + (55 + 44)\\
(iv)\ & (21 + 18) + 72 &&= 21 + (18 + 72)\\
(v)\ & (15 + 27) + 108 &&= 15 + (27 + 108).
\end{aligned}$$

We may remark that the use of the word *Associative* for the description of the property in question is suggested by the fact that it refers to different manners of association of the numbers on the two sides of the equality sign.

We may also note that, it is in view of this property of addition that we can talk of the sum of any three natural numbers a, b, c in the form

$$a + b + c,$$

this expression being equal to any one of the two sums

$$a + (b + c) \text{ or } (a + b) + c$$

which are equal to each other.

EXERCISES

Name the property of addition on the basis of which the following statements are true

$$\begin{aligned}
(i)\ & 7 + (5 + 3) = (7 + 5) + 3\\
(ii)\ & (9 + 21) + 5 = 9 + (21 + 5)
\end{aligned}$$

(*iii*) $13 + (11 + 8) = (13 + 11) + 8$
(*iv*) $(8 + 35) + 27 = 8 + (35 + 27)$.

2. Put down some statements which are true because of the associative property of addition.

3. Each of the following is a true statement. What is the natural number y in each case ?

(*i*) $(13 + 17) + 5 = 13 + (17 + y)$
(*ii*) $19 + (y + 2) = (19 + 13) + 2$
(*iii*) $(15 + y) + 17 = 15 + (11 + 17)$
(*iv*) $(7 + 4) + y = 7 + (4 + 12)$
(*v*) $y + (7 + 3) = (13 + 7) + 3$
(*vi*) $17 + (24 + 11) = (y + 24) + 11$.

Simultaneous Use of the Commutative and Associative Properties. It often happens that in any one given single problem, we use both the Commutative and the Associative properties. We give below two examples to show how with the help of these properties, we can sometimes perform certain calculations more easily than otherwise. In other words, these are the examples of the use of short cuts based on the use of these properties. This use of the laws makes the computation process often simpler. The abbreviation *CA*, will indicate that we have employed the commutative property of addition and the abbreviation *AA*, will indicate our having employed the associative property of addition. Thus, *CA* and *AA* will denote Commutativity of addition and Associativity of additon respectively.

Examples

1 $(25 + 37) + 75 = (37 + 25) + 75$ *CA*
$= 37 + (25 + 75)$ *AA*
$= 37 + 100$
$= 100 + 37 = 137$ *CA*

2. $(9 + 380) + 11 = (380 + 9) + 11$ *CA*
$= 380 + (9 + 11)$ *AA*
$= 380 + 20$
$= 400$.

Note. In actual practice the transformations based on the two addition properties are made mentally.

EXERCISES

1. Compute the following sums through short cuts pointing out the particular property of addition you are employing at each step.

(*i*) $(9 + 48) + 1$ (*ii*) $9 + (380 + 11)$
(*iii*) $(12 + 431) + 88$ (*iv*) $75 + (633 + 25)$
(*v*) $37 + (63 + 24)$ (*vi*) $57 + (28 + 143)$
(*vii*) $14 + (36 + 8)$ (*viii*) $146 + (7 + 24)$.

2. Each of the following is a true statement. What is the natural number x in each case? Justify your answer on the basis of the commutative and associative properties.

(i) $(7 + x) + 11 = (11 + 7) + 8$ (ii) $(5 + 3) + x = (4 + 3) + 5$

(iii) $(15 + x) + 11 = 11\ (14 + 15)$ (iv) $(x + 24) + 11 = 7 + (24 + 11)$

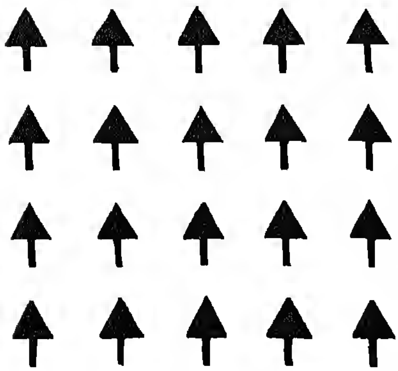
(v) $(23 + x) + 17 = (17 + 9) + 23$ (vi) $(17 + 14) + x = (9 + 14) + 17$.

3. Simplify each of the following in two ways adding from left to right and from right to left. Also justify on the basis of the two laws of addition your getting the same answer in either way.

(i) $15 + 17 + 18$ (ii) $25 + 29 + 38$

(iii) $87 + 59 + 63$ (iv) $77 + 99 + 33$.

Multiplication Composition. Let us consider four rows of trees, each row containing as many as five trees. The total number of trees, counting them by rows will be $5 + 5 + 5 + 5$ which we agree to write as 4×5. Thus, given two natural numbers 4 and 5, we associate to the pair a natural number 4×5, called the product of 4 and 5 taken in this order. In general, if we started with a rows of trees each having b trees, then the number of trees will be $b + b + \ldots$ (a times) which we write as $a \times b$. Thus, to the ordered pair of natural numbers a and b, we associate a natural number



$$a \times b$$

called the product of a and b in this order.

We say that the *multiplication composition* has been defined in the set of natural numbers inasmuch as to any two natural numbers a, b, taken in this order, we can associate a unique natural number called their product and denoted symbolically by

$$a \times b.$$

In the case of letters denoting the numbers, we may write the product of a and b in any one of the following ways

$$a \times b,\ a \,.\, b,\ ab.$$

Properties of Multiplication. Exactly analogous to the two properties of addition, we have the Commutative and Associative properties of multiplication which we shall now describe.

Commutative Property of Multiplication. In the example of trees above, we could have as well counted the trees by columns. There are five columns and each has four trees in it.

The total number, therefore, will be

$$4 + 4 + 4 + 4 + 4 = 5 \times 4.$$

The number of trees being, of course, the same, we have

$$4 \times 5 = 5 \times 4$$

so that we have an instance of the Commutative property of multiplication which we state as follows :

COMMUTATIVE PROPERTY OF MULTIPLICATION

$$a \times b = b \times a$$

for all natural numbers a, b.

EXERCISES

1. Name the property of multiplication, on the basis of which the following statements are true.

(*i*) $15 \times 13 = 13 \times 15$ (*ii*) $8 \times 24 = 24 \times 8$

(*iii*) $107 \times 43 = 43 \times 107$ (*iv*) $7 \times 48 = 48 \times 7$

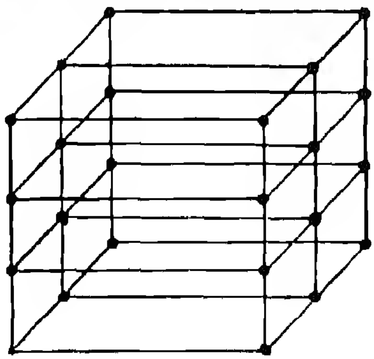
2. Give some statements which are true because of the commutative property of multiplication.

3. Each of the following is a true statement. What is the natural number x in each case ?

(*i*) $x \times 73 = 73 \times 24$ (*ii*) $29 \times x = 69 \times 29$

(*iii*) $33 \times 47 = x \times 33$ (*iv*) $61 \times 72 = 72 \times x$.

Associative Property of Multiplication. Consider a rectangular framework with three beads attached to two of its parallel sides. Further suppose that we have four such frameworks placed one above the other so that we have a structure of the type shown in the diagram We are interested in the total number of beads.



The total number of beads in each horizontal framework is

$$3 \times 2.$$

The total number of horizontal frameworks being 4, we see that the total number of beads is

$$(3 \times 2) \times 4.$$

Again we see that the entire structure can as well be thought of as consisting of 3 vertical frames.

The number of beads in each of the three vertical frames being 2×4, we see that the total number of beads is

$$3 \times (2 \times 4).$$

The number of beads in the entire framework being independent of the manner we proceed with, we see that

$$(3 \times 2) \times 4 = 3 \times (2 \times 4).$$

This is a specific instance of the law of associativity of multiplication, which we formally state as follows :

ASSOCIATIVE PROPERTY OF MULTIPLICATION

$$a \times (b \times c) = (a \times b) \times c$$

for all natural numbers a, b, c.

Note. The commutative and associative properties of multiplication will be referred to as *CM*, *AM* respectively.

EXERCISES

1. Name the property of multiplication on the basis of which the following statements are true.

(*i*) $(17 \times 9) \times 13 = 17 \times (9 \times 13)$.
(*ii*) $(25 \times 4) \times 107 = 25 \times (4 \times 107)$.
(*iii*) $(37 \times 24) \times 7 = 37 \times (24 \times 7)$.
(*iv*) $(102 \times 5) \times 37 = 102 \times (5 \times 37)$.

2. Give some statements which are true because of the associativity of multiplication.

3. Each of the following is a true statement. What is x in each case ?

(*i*) $(7 \times x) \times 20 = (7 \times 25) \times 20$.
(*ii*) $(x \times 13) \times 17 = 21 \times (13 \times 17)$.
(*iii*) $(24 \times 103) \times x = 24 \times (103 \times 9)$.
(*iv*) $(33 \times 3) \times 13 = 33 \times (3 \times x)$.
(*v*) $(45 \times 7) \times 4 = 45 \times (x \times 4)$.
(*vi*) $(111 \times 3) \times 7 = x \times (3 \times 7)$.

Examples

Compute the following products employing short cuts based on the properties of multiplication.

(*i*) $(25 \times 37) \times 4 = (37 \times 25) \times 4$ *CM*
$= 37 \times (25 \times 4)$ *AM*
$= 37 \times 100 = 3700$

(*ii*) $125 \times (77 \times 8) = 125 \times (8 \times 77)$ *CM*
$= (125 \times 8) \times 77$ *AM*
$= 1000 \times 77$
$= 77000.$

EXERCISES

1. What is the number x in respect of the following true statements ? Justify your answer on the basis of the use of the commutative and associative properties.

(*i*) $(x \times 49) \times 37 = (49 \times 37) \times 27.$
(*ii*) $(x \times 13) \times 17 = (13 \times 27) \times 17.$
(*iii*) $(25 \times x) \times 4 = 25 \times (5 \times 4).$
(*iv*) $(23 \times x) \times 4 = (4 \times 23) \times 15.$
(*v*) $(43 \times x) \times 5 = 43 \times (5 \times 13).$
(*vi*) $(x \times 5) \times 3 = (5 \times 3) \times 4.$

2. Compute the following using short cuts.

(*i*) $(2 \times 38) \times 5$ (*ii*) $(5773 \times 5) \times 20$
(*iii*) $(20 \times 84) \times 5$ (*iv*) $50 \times (40 \times 812)$
(*v*) $2 \times (12 \times 15)$ (*vi*) $15 \times (2 \times 13)$
(*vii*) $(4 \times 41) \times 15$ (*viii*) $(25 \times 33) \times 2.$

Multiplication Property of the Number One. In the illustration of trees at the beginning of this section, if we had only one row, the total number of trees would have been five, so that we shall have $1 \times 5 = 5$. Also considering the trees in the row to be in five columns each having one tree, we get $5 \times 1 = 5$. This is one specific instance of the property of the number 1 according to which

$$a \times 1 = a$$

for all natural numbers a.

Briefly, we shall be refering to this property as *MO*. Because of this property of 1, we often refer to the natural number 1 as the *Identify* for multiplication or the *neutral number* for multiplication inasmuch as a number remains unaltered on being multiplied by 1

Note. *Identity for addition or the neutral number for addition.* Since we have not included zero in the set of natural numbers, we are not able to assert that the set of natural numbers admits of an identity or neutral number for addition.

Distributive Law. Suppose that two persons Krishan Lal and Ram Singh have been employed to do a certain job, While Krishan Lal is to be paid rupees five per day, Ram Singh is to receive rupees seven a day. If both work for eight days, we are interested in the total amount to be paid to them.

Now there are two ways of looking at the problem. We may compute the amount received by both in one day and then the total amount in eight days. Or we compute the amount received by each separately in eight days and then the total amount received by both.

(*i*) The amount in ruppees received by both in one day

$$= 5 + 7.$$

The total amount in rupees received by both in 8 days

$$= 8 \times (5 + 7).$$

(*ii*) The amount in rupees received by Krishan Lal in 8 days.

$$= 8 \times 5.$$

The amount in rupees received by Ram Singh in 8 days

$$= 8 \times 7.$$

The total amount in rupees received by both in 8 days

$$= 8 \times 5 + 8 \times 7.$$

Thus, we have

$$8 \times (5 + 7) = 8 \times 5 + 8 \times 7.$$

Surely, the chain of thought involved in the above arguments would be valid if instead of the natural numbers

$$8, 5, 7$$

we have any three natural numbers a, b, c.

Thus, we have the

DISTRIBUTIVE LAW

$$a(b + c) = ab + ac$$

for all natural numbers a, b, c.

It may be seen that while on the left we multiply the sum $(b + c)$ with a, on the right we are separately multiplying b and c with a. We may say, therefore, that the sum has been distributed.

Essentially, therefore, what the law states is that *multiplication distributes addition*. However, we shall refer to this law as only the *Distributive Law* and designate the same as '*D*'.

EXERCISES

1. State the property of natural numbers on the basis of which the following are true statements.

(*i*) $3 \times (5 + 7) = 3 \times 5 + 3 \times 7$
(*ii*) $29 \times (3 + 7) = 29 \times 3 + 29 \times 7$
(*iii*) $11 \times (39 + 21) = 11 \times 39 + 11 \times 21$
(*iv*) $9 \times (17 + 15) = 9 \times 17 + 9 \times 15.$

2. Give some statements which are true because of the distributive law.

3. Each of the following is a true statement. What is the natural number x in each case ?

(*i*) $x(5 + 11) = 13 \times 5 + 13 \times 11$
(*ii*) $5(x + 7) = 5 \times 23 + 5 \times 7$
(*iii*) $9(3 + x) = 9 \times 3 + 9 \times 27$
(*iv*) $23(17 + 13) = x \times 17 + x \times 13$
(*v*) $29(41 + 49) = 29 \times x + 29 \times 49$
(*vi*) $25(51 + 9) = 25 \times 51 + 25 \times x.$

4. Each of the following is a true statement. Give the natural number x in each case. Justify your answer on the basis of the use of the properties.

(*i*) $4 + (x + 5) = 5 + (4 + 9)$
(*ii*) $5 \times (3 + x) = (5 \times 7) + (3 \times 5)$
(*iii*) $x + 39 = 39 + 27$
(*iv*) $13 + (x + 25) = (13 + 109) + 25$
(*v*) $x\,(17 + 3) = (4 \times 17) + (4 \times 3)$
(*vi*) $(x + 5)\,13 = (13 \times 5) + (15 \times 13)$
(*vii*) $(15 + 7)\,x = (22 \times 15) + (7 \times 22)$
(*viii*) $3\,(x + 7) = (7 \times 3) + (9 \times 24)$
(*ix*) $x\,(9 + 3) = (4 \times 27) + (2 \times 18)$.

Example

Simplify the following through short cuts suggested by the distributive law and other properties.

(*i*) $986 \times 693 + 693 \times 14$

(*ii*) $99 \times 99 + 99$

(*iii*) $18 \times 98 + 36$.

Solution

(*i*) $986 \times 693 + 693 \times 14$

$$\begin{aligned} &= 693 \times 986 + 693 \times 14 && CM \\ &= 693 \times (986 + 14) && D \\ &= 693 \times 1000 \\ &= 693000. \end{aligned}$$

(*ii*) $99 \times 99 + 99$

$$\begin{aligned} &= 99 \times 99 + 99 \times 1 && MO \\ &= 99 \times (99 + 1) && D \\ &= 99 \times 100 \\ &= 9900 \end{aligned}$$

(*iii*) $18 \times 98 + 36$

$$\begin{aligned} &= 18 \times 98 + 18 \times 2 \\ &= 18 \times (98 + 2) && D \\ &= 18 \times 100 \\ &= 1800. \end{aligned}$$

EXERCISES

1. Simplify the following, using short cuts.

(*i*) $29 \times 54 + 46 \times 29$
(*ii*) $26 \times 37 + 37 \times 74$
(*iii*) $8 \times 37 + 69 \times 37 + 37 \times 3$
(*iv*) $43 \times 4 + 5 \times 86 + 2 \times 129$
(*v*) $15 \times 19 + 3 \times 35$
(*vi*) $41 \times 8 + 123 \times 4$.

2. Using various properties of addition and multiplication, compute the following mentally.

(i) $(28 + 37) + 72$	(ii) $(25 + 27) + 25$
(iii) $(20 \times 67) \times 5$	(iv) $50 \times (37 \times 20)$
(v) $(23 \times 47) + 47 \times 177$	(vi) $(99 \times 137) + 137$
(vii) $(89 \times 99) + 89$	($viii$) $36 + (28 + 124)$
(ix) $(87 \times 9) + (11 \times 87)$	(x) $(999 \times 999) + 999$
(xi) $(9999 \times 9999) + 9999$	(xii) $(9374 \times 837) + (837 \times 626)$.

3. POWERS, MULTIPLES, RADICALS

Consider any natural number, say 2, and the expressions $2, 2 \times 2, 2 \times 2 \times 2, 2 \times 2 \times 2 \times 2,$. each of which is a product of two with itself taken severel times In different expressions the number 2 occurs a different number of times.

We find it convenient to adopt the following notation to exhibit these different products.

$$
\begin{aligned}
2 &= 2^1 \\
2 \times 2 &= 2^2 \\
2 \times 2 \times 2 &= 2^3 \\
2 \times 2 \times 2 \times 2 &= 2^4 \\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots
\end{aligned}
$$

In general $$\underbrace{2 \times 2 \times 2 \times 2 \ldots\ldots \times 2}_{m\text{-times}} = 2^m.$$

Each of $2^1, 2^2, 2^3, 2^4, 2^5, \ldots \ldots 2^m \ldots\ldots$

is called a **Power** of 2. In particular 2^m is called the m—th power of 2, m is called the **Index** of this power and 2 the **Base** of this power.

Thus, 4 is the fourth power of 2 ; 4 being its index and 2, the base.

We may, however, take any natural number in place of the natural number 2.

Thus, for example, we have

$$3^4 = 3 \times 3 \times 3 \times 3 = 81$$

so that we could say that 81 is the fourth power of 3. Also we have

$$4^3 = 4 \times 4 \times 4 = 64$$

so that 64 is the third power of 4.

Definition. *If a and m are any two natual numbers, then*

$$\underbrace{a \times a \ldots\ldots\ldots\ldots \times a}_{m\text{-times}} = a^m$$

is called the m-th power of a, where m is the index and a is the base of the power.

EXERCISES

Compute the following :

(*i*) 5^3 (*ii*) 6^2

(*iii*) 4^4 (*iv*) 7^3

(*v*) 5^4 (*iv*) 9^2

(*vii*) 10^4 (*viii*) Second power of 5

(*ix*) 6th power of 3 (*x*) 8th power of 2

(*xi*) 5th power of 20 (*xii*) third power of 6

Examples

Simplify :

(*i*) $2^3 \times 2^2$ (*ii*) $x^4\ y^4\ x$.

Solution

(*i*)
$$2^3 = 2 \times 2 \times 2 \text{ and } 2^2 = 2 \times 2$$
$$\therefore \quad 2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2)$$
$$= 2 \times 2 \times 2 \times 2 \times 2$$
$$= 2^5$$

(*ii*) x^2y^4x

$$= x^2\,(y^4x)$$
$$= x^2\,(xy^4) \qquad (y^4 \text{ is treated as one number}),$$
$$= (x^2x)\,y^4$$
$$= x^3y^4. \qquad \text{Definition of } x^3.$$

EXERCISES

1 Simplify the following, x, y being arbitrary natural numbers.

(*i*) $7^2 \times 7^4$ (*ii*) $x^3 \,.\, x^4$

(*iii*) $x \,.\, x^3$ (*iv*) $xy\ x^2$

(*v*) yx^4y^5 (*vi*) $x \,.\, y \,.\, x^2y^3$

(*vii*) $x^2y^3x^7$ (*viii*) $5^3 \times 5^3$

(*ix*) $3^2 \times 3^4$ (*x*) 7×7^6

(*xi*) $5^2 \times 5^5$ (*xii*) $8^2 \times 8^2$.

2. Compute the following :

(*i*) $2^2 \times 3^2$ (*ii*) $2^2 \times 3^3$

(*iii*) $5^3 \times 4^2$ (*iv*) $2^2 \times 10^3$

Multiples

Consider any natural number a and the expressions

$$a,\ a+a,\ a+a+a,\ a+a+a+a+,\ \ldots\ldots$$

each of which is a sum of a with itself taken several times. According to our definition of multiplication, they will be written as

$$1a\ \ 2a,\ 3a,\ 4a,\ \ldots\ldots$$

In general $\underbrace{a+a+a+\ldots\ldots\ldots\ldots+a}_{m\text{-times}} = ma.$

Each of

$$1a, 2a, 3a, \ldots, ma, \ldots$$

is called a multiple of a. Thus, ma is a *multiple of* a. The natural numbers m and a are also referred to as factors of ma.

In the following, the reader can easily see, how the concepts of the power and multiple behave in a similar way.

Multiples	*Powers*
$a = 1a$	$a = a^1$
$a + a = 2a$	$a \times a = a^2$
$a + a + a = 3a$	$a \times a \times a = a^3$
$a + a + a + a = 4a$	$a \times a \times a \times a = a^4$
$\underbrace{a + a + \ldots + a}_{m\text{-times}} = ma$	$\underbrace{a \times a \times a \times a \ldots \times a}_{m\text{-times}} = a^m$
$2a + 3a = a + a + a + a + a$ $= 5a = (2 + 3)\, a$	$a^2 \times a^3 = a \times a \times a \times a \times a$ $= a^5 = a^{2+3}$.

Example. Simplify :

$$2a + 7b + 3a.$$

We have $2a + 7b + 3a$

$$\begin{aligned} &= 2a + (7b + 3a) && AA \\ &= 2a + (3a + 7b) && CA \\ &= (2a + 3a) + 7b && AA \\ &= (a.2 + a.3) + 7b && CM \\ &= a(2 + 3) + 7b && D \\ &= 5a + 7b && CM \end{aligned}$$

EXERCISES

Simplify the following, where a, b, x, y, z are all natural numbers.

(*i*) $5a + 7a$ (*ii*) $a + 3a + 2a$

(*iii*) $5a + 3b + a$ (*iv*) $2a + 4b + a + 3b$

(*v*) $3b + 2a + a + 5b$ (*vi*) $x + y + z + x$.

Radicals

We have seen that the n-th power of any natural number a is again a natural number. If we write the power as b, then we have

$$a^n = b\,;\ a, b, n \text{ being natural numbers.}$$

This statement is written as

$$a = \sqrt[n]{b}$$

and we say that

a is the n-th root of b

if b is the n-th power of a.

The expression

$$\sqrt[n]{b}$$

is also called **a radical.**

For example, we have

$$\sqrt[2]{4} = 2 \qquad \text{because} \quad 2^2 = 4$$

$$\sqrt[3]{27} = 3 \qquad \text{because} \quad 3^3 = 27$$

$$\sqrt[4]{10000} = 10 \qquad \text{because} \quad 10^4 = 10000.$$

It is usual to call the *2-th* root of a number as its square root. Also the 2-th root $\sqrt[2]{a}$ of a is simply written as $\sqrt{a}$ without the symbol 2 so that $\sqrt{a}$ denotes what is called the square root of a.

EXERCISES

Simplify by trial the following radicals.

(*i*) $\sqrt[3]{8}$ (*ii*) $\sqrt[5]{32}$

(*iii*) $\sqrt{81}$ (*iv*) $\sqrt[3]{64}$

(*v*) $\sqrt[6]{64}$ (*vi*) $\sqrt[4]{256}$

(*vii*) $\sqrt[3]{125}$ (*viii*) $\sqrt{36}$

(*ix*) $\sqrt{16}$ (*x*) $\sqrt[3]{216}$.

Whereas every power of a natural number is a natural number, we must see that a natural number need not essentially be the power of a natural number and as such the n-th root of a natural number may not be always meaningful. For example, the number 2 is not the 2-th power of any natural number and so we cannot talk of the square root of the number 2, in respect of the set of natural numbers.

EXERCISES

1. Which of the following numbers are squares of natural numbers? Wherever possible, find the square roots.

(*i*) 1 (*ii*) 4 (*iii*) 7

(*iv*) 9 (*v*) 12 (*vi*) 16

(*vii*) 36 (*viii*) 48 (*ix*) 49

(*x*) 121.

2. Which of the following numbers are cubes of natural numbers? Wherever possible, find the cube roots.

(*i*) 8 (*ii*) 16 (*iii*) 32

(*iv*) 27 (*v*) 48

3. Which of the following numbers are fourth powers of natural numbers? Wherever possible, find the fourth roots.

(*i*) 16 (*ii*) 32 (*iii*) 64

(*iv*) 81 (*v*) 625.

[**Hints.** 1. (x) 121 is the square of 11

i.e. $11^2 = 121$

$\therefore \sqrt{121} = 11$

2 (iii) 32 is not the cube of any natural number and as such the expression $\sqrt[3]{32}$ is meaningless in respect of the set of natural numbers

3. (iv) 81 is the fourth power of 3, so that $3^4 = 81$.

$\therefore \sqrt[4]{81} = 3.$]

4. Simplify the following, a, b, c being natural numbers.

(i) $\sqrt{a^2}$ (ii) $\sqrt{a^2b^2}$

(iii) $\sqrt{a^4b^4}$ (iv) $\sqrt[3]{b^3}$

(v) $\sqrt[3]{a^6b^3}$ (vi) $\sqrt[3]{8a^3}$

(vii) $\sqrt{a^4}$ (viii) $\sqrt{16a^4b^4}$

(ix) $\sqrt{81b^8c^{12}}$

Importance of the Distributive Law

The ditributive law, *viz.*,

$$a(b + c) = ab + ac$$

will play two roles

Firstly it expresses the product $a(b + c)$ as a sum *viz.*, $ab + ac$.

Conversely, it can be thought of as expressing the sum $ab + ac$ as a product. Both these roles of the distributive law will be illustrated in the following examples and exercises. It may be remembered that, for expressing any sum as a product, we look for a factor of each of the terms constituting the sum. For example, in the expression

$$ax + bx + cx,$$

we see that x is common in each of the three parts of the sum. We have

$$\begin{aligned} ax + bx + cx &= (ax + bx) + cx \\ &= (a + b)x + cx \\ &= [(a + b) + c]x \\ &= (a + b + c)x \\ &= x(a + b + c). \end{aligned}$$

Examples

1. Show that

$$[(a + b) + c] + d = [(d + b) + a] + c$$

for all natural numbers a, b, c, d.

Proof. We have

$$\begin{aligned} [(a + b) + c] + d &= (a + b) + (c + d) && AA \\ &= (a + b) + (d + c) && CA \\ &= [(a + b) + d] + c && AA \\ &= [a + (b + d)] + c && AA \\ &= [a + (d + b)] + c && CA \\ &= [(d + b) + a] + c && CA \end{aligned}$$

2. For all natural numbers a, b, c, prove that

$$(a+b)\,c = ac + bc.$$

Proof. We have

$$\begin{aligned}(a+b)\,c &= c(a+b) && CM\\ &= ca + cb && D\\ &= ac + bc. && CM\end{aligned}$$

3. For all natural numbers a, b, c, d, show that

$$a(b+c+d) = ab + ac + ad.$$

Proof. We have

$$\begin{aligned}a(b+c+d) &= a[(b+c)+d] && AA\\ &= a(b+c) + ad && D\\ &= (ab+ac) + ad && D\\ &= ab + ac + ad. && AA\end{aligned}$$

4. For all natural numbers a, b, c, d, show that

$$(a+b)\,(c+d) = ac + ad + bc + bd$$

Proof. Treating $(c+d)$ as one number, and using the result of example 2 above, we have

$$\begin{aligned}(a+b)\,(c+d) &= a(c+d) + b(c+d)\\ &= (ac+ad) + (bc+bd) && D\\ &= ac + ad + bc + bd. && AA\end{aligned}$$

5 Prove that

$$(a+b)^2 = a^2 + 2ab + b^2$$

where a and b are any natural numbers.

Proof. We have

$$\begin{aligned}(a+b)^2 &= (a+b)\,(a+b) && \textit{Definition}\\ &= (a+b)\,a + (a+b)\,b && D\\ &= a(a+b) + b(a+b) && CM\\ &= (a\,.\,a + a\,.\,b) + (b\,.\,a + b\,.\,b) && D\\ &= (a^2 + ab) + (ba + b^2)\\ &= (a^2 + ab) + (ab + b^2) && CM\\ &= [(a^2 + ab) + ab] + b^2 && AA\\ &= [a^2 + (ab + ab)] + b^2\\ &= (a^2 + 2ab) + b^2 && \textit{Definition}\\ &= a^2 + 2ab + b^2 && AA\end{aligned}$$

Note. In each of the examples above, we have tried to justify every step in terms of the basic properties. Once the reader becomes familiar with this justification, he is supposed to take note of the laws in operation mentally to arrive at the final results. He may, as he gains practice, even learn to skip over some steps.

EXERCISES

1. For all natural numbers a, b, c, d, prove that

(i) $(a+b)+(c+d) \quad = (a+c)+(d+b)$

(*ii*) $(a + b) + (c + d) = (b + c) + (d + a)$

(*iii*) $(2a + 3b) + (5a + 4b) = 7(a + b)$

(*iv*) $(11a + 13b) + (a + 4b) = 12a + 17b.$

2. a, b, c, d, x, y are any natural numbers ; show that :

(*i*) $x (a + b + c) = ax + bx + cx$

(*ii*) $(a + b + c) d = ad + bd + cd$

(*iii*) $(2a + 5b) (4x + 3y) = 8ax + 20bx + 6ay + 15 by$

3. a, b, c, x, y, z are any natural numbers ; show that :

(*i*) $(x + 2) (x + 3) = x^2 + 5x + 6$

(*ii*) $(x + a) (x + b) = x^2 + (a + b) x + ab$

(*iii*) $(3x + 1) (2x + 5) = 6x^2 + 17 x + 5$

(*iv*) $(x + 5y) (x + 8y) = x^2 + 13xy + 40y^2$

(*v*) $(2x + 3y) (7x + 10y) = 14x^2 + 41xy + 30y^2$

(*vi*) $(x + 1) (x + 2) (x + 3) = x^3 + 6x^2 + 11x + 6$

(*vii*) $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$

(*viii*) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(*ix*) $x (x^2 + 3) = x^3 + 3x$

(*x*) $(x + 2) (2x^2 + 5) = 2x^3 + 4x^2 + 5x + 10$

(*xi*) $y^2 (xy + x^2) = x^2y^2 + xy^3$

(*xii*) $x (y + zx + x^3) = xy + x^2z + x^4$

(*xiii*) $(a + b) (x + y + z) = ax + bx + ay + by + az + bz$

Examples

Express the following sums as products :

(*i*) $ax + bx$

(*ii*) $ax + bx + cx$

(*iii*) $x^2y + xy^2$

(*iv*) $ax + bx + (a + b) y$

whhre a, b, c, x, y are all natural numbers.

Solution

(*i*) We have

$$\begin{aligned} ax + bx &= xa + xb && \textit{CM} \\ &= x (a + b). && \textit{D} \end{aligned}$$

(*ii*)
$$\begin{aligned} ax + bx + cx &= xa + xb + xc && \textit{CM} \\ &= (xa + xb) + xc && \textit{AA} \\ &= x (a + b) + xc && \textit{D} \\ &= x [(a + b) + c] && \textit{D} \\ &= x (a + b + c). && \textit{AA} \end{aligned}$$

(*iii*)
$$\begin{aligned} x^2y + xy^2 &= (x \,.\, x) y + x (y.\, y) && \\ &= x (xy) + (xy) y && \textit{AM} \\ &= (xy) x + (xy) y && \textit{CM} \\ &= xy (x + y). && \textit{D} \end{aligned}$$

$$\begin{aligned} (iv)\ ax + bx + (a+b)\,y &= xa + xb + (a+b)\,y && CM \\ &= x\,(a+b) + (a+b)\,y && D \\ &= (a+b)\,x + (a+b)\,y && CM \\ &= (a+b)\,(x+y) && D \end{aligned}$$

EXERCISES

Using the distributive law and other properties, express the following sums as products.

(*i*) $3x + 3y$ (*ii*) $ax + ya$
(*iii*) $2xy + 2xz$ (*iv*) $3xy + 6xz$
(*v*) $4xy + 5xz$ (*vi*) $2\,(a+b)\,x + 3\,(a+b)\,y$
(*vii*) $8y + 6xy$ (*viii*) $a^2 + 2a$
(*ix*) $x^3 + x^2$ (*x*) $3x^2y + 5xy^2$
(*xi*) $3x + 3y + 3z$ (*xii*) $3ax + 6by + 9c$
(*xiii*) $xyz + x^2y + x^2z$ (*xiv*) $3x + 3y + 5\,(x+y)$
(*xv*) $2x + 6y + 3a\,(x+3y)$ (*xvi*) $2ax + 5ay + b\,(2x+5y)$
(*xvii*) $ax + by + ay + bx$ (*xviii*) $6xy + 8x + 9y + 12$
(*xix*) $4ab + 16a + b + 4$ (*xx*) $xy + x + y + 1$
(*xxi*) $2u + 2v + uv + 4$ (*xxii*) $ax + ay + xy + a^2$.

4. ORDER RELATION

Besides the addition and multiplication compositions in the set of natural numbers, we also have in the set what is called an *Order Relation* which shall be referred to as the

'*is greater than*'

relation and symbolically represented as

'$>$'.

While the addition and multiplication compositions associate to each pair of natural numbers a, b, the natural numbers denoted respectively by $a + b$, ab, the order relation denoted by $>$, is such that given any two *different* natural numbers a, b, we have

either $a > b$ or $b > a$.

In the following we shall discuss a few basic properties of this order relation.

Starting from the natural number, 1, we successively obtain all the natural numbers by adding 1, to the natural number obtained at any stage Thus we have,

$$2 = 1 + 1,\ 3 = 2 + 1,\ 4 = 3 + 1,\ 5 = 4 + 1, \ldots\ \ldots$$

We say that the number at each stage *is greater than* each of the numbers obtained in the earlier stages. Thus, for instance,

(*i*) 2 is greater than 1 or $2 > 1$.
(*ii*) 4 is greater than 2 or $4 > 2$.
(*iii*) 11 is greater than 6 or $11 > 6$.

In general, if a and b are natural numbers such that *a is greater than b*, we write in symbols.

$$a > b.$$

If we start with any two different numbers, say 12 and 17, we have that one of these, 17, can be obtained from 12 by successively adding to 12, the number 1 five times. In other words, we have

$$17 = 12 + 1 + 1 + 1 + 1 + 1$$

so that 17 is greater than 12, *i.e.*, $17 > 12$. Certainly here 12 is not greater than 17. We see, therefore, if a and b are any two different natural numbers, then either

$$a > b \text{ or } b > a.$$

This leads us to state what is called the **Trichotomy Law.** According to this law, given any two natural numbers a and b, we have one and only one of the following *three possiblities.*

(*i*) $a = b$ (*ii*) $a > b$ (*iii*) $b > a$.

In the following we shall see how it is possible to express the relation 'is greater than' in terms of the addition composition.

Now $13 > 10$ and 13 can be obtained from 10 by successively adding to 10, the number 1 three times so that we have

$$13 = 10 + 1 + 1 + 1.$$

This could be re-written as

$$13 = 10 + 3.$$

Thus, $13 > 10$ means that, there exists a natural number 3 such that

$$13 = 10 + 3.$$

In general *if $a > b$ there exists a natural number d such that $a = b + d$.* For example, If $a = 7$, $b = 4$ so that $7 > 4$ we have $d = 3$. If $a = 18$, $b = 12$ so that $18 > 12$ we have $d = 6$. Conversely, if a, b, d are natural numbers such that

$$a = b + d$$

then a can be obtained by adding to b, the number 1 successively d times, so that we have

$$a > b.$$

Combining the two, we have, therefore, $a > b$ if and only if there exits a natural number d such that

$$a = b + d.$$

Note. It may be noted that if

$$a = b + d,$$

then we also have $a > d$ because the number a can as well be obtained by successively adding the number 1 to the number d, b times. Thus, whenever

$$a = b + d,$$

we have

$$a > b \text{ and } a > d.$$

For example $23 = 14 + 9$
gives $23 > 14$ and $23 > 9$.

Example 1. $47 > 35$

because $47 = 35 + 12.$

Example 2. $23 = 19 + 4$

means that $23 > 19.$

EXERCISES

1. Give reasons for the following statements.

(*i*) $13 > 7$ (*ii*) $25 > 12$
(*iii*) $37 > 33$ (*iv*) $50 > 17$
(*v*) $16 > 14$ (*vi*) $19 > 16$

2. Express the following statements symbolically.

(*i*) 12 is greater than 11. (*ii*) 15 is greater than 5.
(*iii*) 24 is greater than 21. (*iv*) 37 is greater than 12

3. Which of the following are true statements and which are false statements ?

(*i*) 5 is greater than 7. (*ii*) 12 is greater than 3.
(*iii*) 31 is greater than 35. (*iv*) 3 is greater than 3.
(*v*) 11 is greater than 8. (*vi*) 13 is greater than 14.
(*vii*) 25 is greater than 21 (*viii*) 29 is greater than 29
(*ix*) $7 > 9$ (*x*) $12 > 4$
(*xi*) $28 > 28$ (*xii*) $16 > 15$
(*xiii*) $41 > 41$ (*xiv*) $48 > 47$
(*xv*) $48 > 49$ (*xvi*) $37 > 1$
(*xvii*) $50 > 50$ (*xviii*) $103 > 127$
(*xix*) $256 > 204$ (*xx*) $1 > 1$

4. In place of each false statement in *Ex.* 3 above write the statement about the two numbers which is true. As for example, the true statement in place of (*iii*) will be 35 is greater than 31.

Basic Laws of the Order Relation. In addition to the *Trichotomy Law* of the order relation '$>$' we have in the following, some other basic laws of the relation.

Transitivity of the Order Relation. Let us suppose that a student has to pay Rs. 7, Rs. 9 or Rs. 12 per month as tuition fee according as he is in the 9th, 10th or 11th class. Now, we know that

$$9 > 7 \quad \text{and} \quad 12 > 9$$

so that a student of the 10th pays more fee than a student of the 9th class and a student of the 11th pays more fee than a student of the 10th class. Certainly, we have that a student of the 11th class pays more fee than a student of the 9th class, as we have

$$12 > 7.$$

We have here a situation which can be described as follows :

$$12 > 9 \text{ and } 9 > 7$$

give

$$12 > 7.$$

Similarly, we have,

$$13 > 8 \text{ and } 8 > 3$$

give

$$13 > 3.$$

These are specific illustrations of the transitivity of the order relation according to which

if $a > b$ and $b > c$

then $a > c$.

This property of the relation '$>$' can be easily seen as follows.

If $a > b$,

there exists a natural number d such that

$$a = b + d. \tag{1}$$

Again, because $b > c$

there exists a natural number e such that

$$b = c + e. \tag{2}$$

(1) and (2) give

$$a = (c + e) + d$$

$$\therefore \quad a = c + (e + d). \qquad A A$$

Thus, there exists a natural number $(e + d)$ such that

$$a = c + (e + d)$$

so that we have $a > c$.

Relationship of the Order Relation with the Addition Composition

Suppose in our illustration of the previous section each student of the school has to pay Rs. 2 per month as Library, Building and other fees. Then a student of the 10th class will have to pay Rs. $(9 + 2)$ and that of the 9th class will pay Rs. $(7 + 2)$ per month. Essentially the 10th class student pays more fees than the 9th class student as we have

$$9 + 2 > 7 + 2.$$

Thus, if $9 > 7$ then $9 + 2 > 7 + 2$.

Similarly if $12 > 9$ then $12 + 2 > 9 + 2$

In general we see that

if $a > b$,

and c is any natural number, then

$$a + c > b + c.$$

In other words, one may say that a true order statement pertaining to two natural numbers, remains true if we add the same natural number to each of the two sides of the symbol $>$.

This property is expressed by saying that *the addition composition is compatible with the order relation.*

If $a > b$ there exists a natural number d such that

$$a = b + d$$

$\therefore$ $\quad a + c = (b + d) + c$

$\therefore$ $\quad a + c = (b + c) + d \qquad$ *AA and CA*

and so $\quad a + c > b + c.$

Thus, we have the following law :

Compatibility of addition composition with order relation. For all natural numbers a, b, c,

if $\quad a > b \quad$ *then* $\quad a + c > b + c.$

Below, we prove two important results which will be frequently used in the context of solutions of equations and inequalities.

Cancellation law for addition. The law states that

if $\quad a + c = b + c \quad$ *then* $\quad a = b$

whatever natural numbers a, b, c may be.

Proof. By the Trichotomy Law, there exists one and only one of the following three possibilities.

(*i*) $a > b$ $\qquad$ (*ii*) $b > a$ $\qquad$ (*iii*) $a = b$

We shall show that the possibilities (*i*) and (*ii*) will lead to a contradiction of the hypothesis.

(*i*) If $\quad a > b \quad$ then $\quad a + c > b + c.$

(*ii*) If $\quad b > a \quad$ then $\quad b + c > a + c.$

But we are given that $a + c = b + c$ so that neither $a + c > b + c$ nor $b + c > a + c$.

$\therefore$ we must have $\quad a = b.$

Theorem. If $a + c > b + c$, then $a > b$ whatever natural numbers a, b, c may be.

Proof. By the Trichotomy Law, there exists one and only one of the three possibilities.

(*i*) $a > b$ $\qquad$ (*ii*) $b > a$ $\qquad$ (*iii*) $a = b$

We shall see that the possibilities (*ii*) and (*iii*) will lead to contradictions of the hypothesis

$$a + c > b + c.$$

(*i*) If $\quad b > a \quad$ then $\quad b + c > a + c.$

(*ii*) If $\quad a = b \quad$ then $\quad a + c = b + c.$

But we are given that $a + c > b + c$ so that we have neither $b + c > a + c$ nor $a + c = b + c$.

$\therefore$ we must have $a > b$.

As a consequence of our study of the relationship of $>$ with addition, we have the following fundamental results.

(*i*) A true statement of equality continues to be true if we cancel from both sides of the symbol '=' the same natural number occuring as an addend. Thus, for example we have

if $\quad x + 5 = 11 \quad$ then $\quad x = 6$

because, we can rewrite 11 as $6 + 5$.

Note. We also have

if $\quad x = 6 \quad$ then $\quad x + 5 = 6 + 5 = 11$.

(*ii*) A true statement about the relation '>' continues to be true if we add to both sides of the symbol the same natural number or if we cancel from both sides the same natural number occuring as an addend. For example,

if $\quad (2x + 7) > (x + 9) \quad$ then $\quad x > 2$.

We have

$$(2x + 7) > (x + 9)$$

$\because \quad x + (x + 7) > (x + 7) + 2$

$\therefore \quad x + (x + 7) > 2 + (x + 7)$

and cancelling $(x + 7)$ on both sides we have

$$x > 2.$$

Conversely, if $\quad x > 2 \quad$ then $\quad 2x + 7 > x + 9$.

We have

if $\quad x > 2 \quad$ then $\quad x + 7 > 2 + 7 = 9$

and so $\quad x + (x + 7) > x + 9$

which gives $\quad 2x + 7 > x + 9$.

EXERCISES

x is any natural number. Prove that

(*i*) if $7x + 23 = 2x + 38$ then $5x = 15$

(*ii*) if $2x = 1$ then $7x + 9 = 5(x + 2)$

(*iii*) if $x + 3 > 14$ then $x > 11$

(*iv*) if $x > 3$ then $4x + 5 > 3x + 8$

(*v*) if $5x + 3 > 2x + 5$ then $3x > 2$.

Relationship of the Order Relation with the Multiplication Composition. Analogous to the law of compatibility of the addition composition with the order relation, we have the law of compatibility of the multiplication composition with the order relation

Suppose we had 36 students in each of the classes 10th and 11th, we were considering. Then the total tuition fee paid by students of the two classes will be rupees 9×36 and 12×36 respectively. The total tuition fee collected from the 11th class will be greater than that collected from 10th class. We, therefore have

if $\quad 12 > 9 \quad$ then $\quad 12 \times 36 > 9 \times 36$.

Similarly, if there were 36 students in the 9th class we would have got the result ·

as $9 > 7$ we have $9 \times 36 > 7 \times 36$.

In general, we see that,

if $\quad a > b \quad$ and c is any natural number, then

$$ac > bc.$$

We say that *the multiplication composition is compatible with the order relation.*

We may see this as follows :

Let $a > b$.

There exists a natural number d such that

$$a = b + d$$

$$\therefore \quad ca = c\,(b + d)$$

$$= cb + cd \qquad DM$$

$$\text{i.e.,} \quad ac = bc + dc \qquad C$$

$$\therefore \quad ac > bc.$$

And so we have the following law :

Compatibility of the multiplication composition with order relation. For natural numbers a, b, c,

if $\quad a > b \quad$ *then* $\quad ac > bc.$

Cancellation Law for Multiplication. Analogous to the cancellation law for addition, we have the concellation law for multiplication according to which,

for all natural numbers a, b, c,

if $\quad ac = bc \quad$ *then* $\quad a = b.$

We see its truth as follows :

We have one and only one of the following three possibilities.

(*i*) $a > b$ $\qquad$ (*ii*) $b > a$ $\qquad$ (*iii*) $a = b$.

In the case (*i*) we have $\quad ac > bc$.

In the case (*ii*) we have $\quad bc > ac$.

But neither $ac > bc$ nor $bc > ac$ because we are given that $ac = bc$.

$\therefore$ we must have $a = b$.

Theorem. Whatever natural numbers a, b, c may be, show that if

$ac > bc \quad$ then $\quad a > b$

Proof. We have one and only one of the following three possibilities.

(*i*) $a > b$ $\qquad$ (*ii*) $b > a$ $\qquad$ (*iii*) $a = b$.

In the case (*ii*) $bc > ac$.

In the case (*iii*) $ac = bc$.

But neither $bc > ac \quad$ nor $\quad ac = bc$.

$\therefore$ we must have $a > b$.

As a consequence of the study of the relationship of $>$ with multiplication, we have the following results of fundamental importance.

(*i*) A true statement of equality continues to be a true statement if we cancel from both sides of the symbol '=' the same natural number occuring as a factor. Thus, for example, we have :

if $\qquad 2x = 18 \qquad$ then $x = 9$

because we can rewrite 18 as 2×9.

Note. We also have : if $x = 9$ then $x \,.\, 2 = 9 \,.\, 2$ *i.e.*, $2x = 18$.

(*ii*) A true statement about the relation $>$ continues to be true if we multiply both sides of the symbol by the same natural number or if we cancel from both sides the same natural number occuring as a factor. For example, we have :

if $\qquad 3x > 21 \qquad$ then $x > 7$

because we could write 21 as 3×7.

Conversely, if $\qquad x > 7$

then $\qquad 3x > 3 \times 7$

i.e., $\qquad 3x > 21$.

EXERCISES

x is any natural number. Prove that :

(*i*) if $x + 3 = 7$	then	$x^2 + 3x = 7x$
(*ii*) if $2x + 5 = x + 11$	then	$6x + 15 = 3x + 33$
(*iii*) if $2x + 1 > x + 4$	then	$6x + 3 > 3x + 12$
(*iv*) if $x > 5$	then	$x^2 > 5x$
(*v*) if $x^3 > x^2 + 3x$	then	$x^2 > x + 3$.

Examples

1. For any natural number x, prove that

if $\qquad 3x + 5 = x + 17 \qquad$ then $\qquad x = 6$.

Solution. $\qquad 3x + 5 = x + 17$.

Writing 17 as $12 + 5$, we have by using *AA*

$$3x + 5 = (x + 12) + 5.$$

Cancelling the addend 5 on both sides, we have

$$3x = x + 12. \qquad (i)$$

Again $\qquad 3x = x + 2x$ and so (*i*) gives

$$x + 2x = x + 12.$$

Cancelling the addend x on both sides, we have

$$2x = 12.$$

Cancelling the factor 2 on both sides, we have

$$x = 6.$$

2. For any natural numbers a and b, show that if

$a > b \qquad$ then $a^2 > b^2$.

Solution. $\qquad a > b$

$\therefore \quad a \,.\, a > a \,.\, b \quad$ i.e., $\quad a^2 > ab \qquad (1)$

Again $\quad a > b$

$\therefore \quad a \,.\, b > b \,.\, b \quad$ i.e., $\quad ab > b^2 \qquad (2)$

From (1) and (2), using transitive property of the relation '$>$' we have

$$a^2 > b^2.$$

3. For any natural numbers a, b, c, d, prove that

if $\quad a > b$ and $c > d \quad$ then $\quad a + c > b + d.$

Solution. $\quad a > b$

$\therefore \quad a + c > b + c \qquad (i)$

Also $\quad c > d$

$\therefore \quad b + c > b + d \qquad (ii)$

From (i) and (ii), we have $\quad a + c > b + d$

Alternative Solution.

There exist natural numbers p and q such that

$$a = b + p \qquad (i)$$
$$c = d + q \qquad (ii)$$

From (i) and (ii), we have

$$\begin{aligned} a + c &= (b + p) + (d + q) \\ &= (b + d) + (p + q) \qquad C\,A,\ A\,A \end{aligned}$$

$\therefore \quad a + c > b + d.$

EXERCISES

1. x is any natural number. Prove that

   (i) if $4x + 1 = 13$ then $x = 3$
   (ii) if $x = 3$ then $4x + 1 = 13$
   (iii) if $4x + 1 > 17$ then $x > 4$
   (iv) if $x > 4$ then $4x + 1 > 17$
   (v) if $17 > 4x + 1$ then $4 > x$
   (vi) if $4 > x$ then $17 > 4x + 1$

2. If a, b, c, d are natural numbers such that

$$a > b \quad \text{and} \quad c > d$$

then $\quad ac > bd.$

3. For any natural numbers a, b, show that

   if $\quad a > b \quad$ then $\quad a^3 > b^3.$

4. a and b are any natural numbers. Show that

   (i) if $a > 2$ then $(5a + 3b) + 14 > 3\,(b + 8)$
   (ii) if $(5a + 3b) + 14 > 3\,(b + 8)$ then $a > 2$
   (iii) if $a > 11$ then $a^2 + 3a + 7 > 7\,(2a + 1)$
   (iv) if $a^2 + 3a + 7 > 7\,(2a + 1)$ then $a > 11.$

Remarks. Very often instead of saying that b is greater than a, we say that a is smaller than b or a is less than b and symbolically write

$$a < b.$$

Thus, for example, $9 < 12$ because $12 > 9$.

5. For any natural numbers a, b, c, show that

(*i*)	if $a < b$	then	$a + c < b + c$
(*ii*)	if $a + c < b + c$,	then	$a < b$
(*iii*)	if $a < b$	then	$ac < bc$
(*iv*)	if $ac < bc$	then	$a < b$
(*v*)	if $a < b$	then	$a^2 < b^2$
(*vi*)	if $a < b$ and $b < c$	then	$a < c$.

6. For natural numbers a, b, c, d, show that

(*i*)	if $a < b$ and $c < d$	then	$a + c < b + d$
(*ii*)	if $a < b$ and $c < d$	then	$ac < bd$.

7. x is any natural number. Prove that

(*i*)	if $5x + 4 < 7$	then	$5x < 3$
(*ii*)	if $5x < 3$	then	$5x + 4 < 7$
(*iii*)	if $2x + 3 < 15$	then	$x < 6$
(*iv*)	if $x < 6$	then	$2x + 3 < 15$.

5. OPEN STATEMENTS

We know of true and false statements. For example each of the following is a true statement.

(*i*) $3 + 4 = 4 + 3$ (*ii*) $(2 \times 3) + 4 = 2 \times (3 + 2)$

(*iii*) $5 + 3 > 4 + 3$ (*iv*) $4 \times 2 < 7 \times 2$

(*v*) If x is a natural number other than 1, then

$$x > 1.$$

Again each of the following is a false statement.

(*i*) $3 > 5$ (*ii*) $3 + 4 = 3 + 2$

(*iii*) $9 < 9$ (*iv*) $7 \times 3 > 8 \times 3$

(*v*) For natural numbers a and b, if $a > b$ then

$$b^2 > a^2.$$

Let us now consider the statement

$$3x + 4 = 13$$

and the question

'Is the statement true ?'

Because of the presence of x, in the statement, a definite answer 'yes' or 'no' is not possible in this case. Because of the cancellation laws of addition and multiplication, we have that

if $3x + 4 = 13$ then $x = 3$,

and conversely, also, we have

if $\qquad x = 3 \qquad$ then $\qquad 3x + 4 = 13,$

We, therefore, see that the statement is true if and only if

$$x = 3$$

and is false if and only if $\qquad x \neq 3.$

The symbol $\neq$ is to be read as 'is not equal to'

Similarly, the statement

$$5x + 17 = 7$$

is true for $x = 2$ and is false when $x \neq 2.$

Statements, about which we cannot straightaway say whether they are true or false are called **Open Statements**, inasmuch as the question of such a statement being true or false is open till we have some more information about the same.

While each of the two open statements considered above is an equation, we also have open statements which are called *Inequations* or *Inequalities*.

Let us consider the open statement

$$3x + 1 < 20$$

which is an inequality. We may feel interested in all those of the *natural numbers* such that if x is replaced by any of them the statement becomes true.

By trial, it is easy to see that these numbers are 1, 2, 3, 4, 5, 6 and no other.

Similarly, we may consider the open statement

$$2x + 1 > 4$$

which is again an inequality. We may feel interested in all those of the *even natural numbers* such that if x were replaced by any of them, the statement becomes true. It is not difficult to see that these even numbers are

$$2, 4, 6, 8, \ldots.$$

In the light of the observations made above, we now introduce the notions of

(*i*) Variable.
(*ii*) Domain of a variable.
(*iii*) Truth set of an open statement.

Each open statement, *equation or inequality*, will involve besides some specific numbers one or more letters. As a member of the alphabet, this letter has no meaning in the context of its being there mixed up with numbers. In fact, each open statement, must have associated with it a *set of numbers* and the letter or letters thought of as any member of this set. Thus, if x be a letter appearing in any open statement and S be any set of numbers such that x could be any member of S we shall call

x as a variable with the set S as its domain.

The set S may thus be as well thought of as the set of *replacements* of x.

The members of S which when substituted for the variable, render the given open statement true, are referred to as *Truth numbers* and the set of all *truth numbers* is called the *Truth set* of the open statement.

For example, with reference to the open statement

$$3x + 1 > 20$$

the domain of the variable x being the set of natural numbers, the truth set has as its members,

$$1, 2, 3, 4, 5, 6.$$

Again in the context of the open statement

$$2x + 1 > 4$$

the domain of the variable x being the set of all even natural numbers

$$2, 4, 6, 8, 10. \ldots$$

the truth set is the domain itself.

Sometimes, we may not be able to find any member of the domain which when subsituted for the variable makes the open statement true. As, for example, the open statement $x < 1$, the domain being the set of all natural numbers, will not be made true whatever natural number we may substitute for x. In such a case we say that the truth set of the open statement is *Empty* or *Void*.

We give below a table showing the truth sets of some open statements, the domain of the variable being always the set of natural numbers.

Open statements	*Truth set* (members)
(*i*) $4x + 1 = 13$	$\{3\}$
(*ii*) $4x + 3 < 19$	$\{1, 2, 3\}$
(*iii*) $2x + 1 = 4$	Void set
(*iv*) $x^2 = 1$	$\{1\}$
(*v*) $2x + 1 > 3$	$\{2, 3, 4, \ldots\}$

The reader is advised to see how the truth sets would alter, if the domain in each case were changed to

(*a*) the set of all even natural numbers.

(*b*) the set of all odd nature numbers.

No doubt, we have tried to only guess the truth sets of open statements given above. But, in general, however, this is not possible. It will perhaps be a matter of relief to see that we have pretty systematic processes, based on the properties of the addition and multiplication compositions and the order relation, to determine the truth sets of open statements given in the form of equations and inequalities.

The process of determining the truth set of an open statement is also referred to as *solving* the open statement and each member of the truth set is called a *root* or *solution* of the open statement. We also say that a member of the truth set of an open statement satisfies the same.

Equivalent Open Statements

The process of determining the truth set of any given open statement consists in obtaining a chain of open statements such that

(*i*) the truth set of each of the open statements in the chain is the same as the truth set of the given open statements ;

(*ii*) each of the open statements in the chain is simpler than each of those open statements comming before it ;

(*iii*) the last of the open statements is such that its truth set is obvious.

We say that two *open statements are equivalent if the truth set of one is the same as that of the other*. The two equivalent statements, thus, have the same truth set. For example, the statements $3x + 5 = 8$ and $3x + 6 = 9$ are equivalent statements.

In order to determine the truth set of an open statement, we obtain a chain of pairs of equivalent open statements such that the truth set of the last is obvious. The truth set of the last of the open statements is also then the truth set of the given open statement. In the present chapter, unless otherwise specified, the domain of the variable will be the set of natural numbers

We should make a note of the following four basic principles for obtaining a chain of equivalent open statements.

(*i*) Adding to both sides of an equation or an inequality the same natural number.

(*ii*) Cancelling the same natural number occuring as an addend on the two sides of an equation or an inequality.

(*iii*) Multiplying both sides of an equation or an inequality by the same natural number.

(*iv*) Cancelling the same natural number occuring as a factor on the two sides of an equation or an inequality.

It is interesting to see that if we go from one open statement to another by the use of the principle (*i*) we can go back to the original open statement using the principle (*ii*). Similarly, if an open statement is obtained from another by using the principle (*iii*), we can always go back to the original open statement using the principle (*iv*). Thus, whenever, an open statement is true, another obtained from it by using any of the four principles will as well be true and *vice versa*.

We illustrate the process with the help of a few examples.

Examples

1. Solve the equation $3x + 5 = 11$.

Solution. The equation can be rewritten in the form

$$3x + 5 = 6 + 5.$$

Cancelling the addend 5 on the two sides we get

$$3x = 6. \qquad \text{(using } ii\text{)}$$

This can again be rewritten in the form

$$3x = 3 \cdot 2.$$

Cancelling the factor 3 on the two sides, we get

$$x = 2. \qquad \text{(using } iv\text{)}$$

The required truth set consists of the number 2.

Verification. As a check or verification we may see that

$$(3 \times 2) + 5 = 11$$

is true.

2. Solve the equation

$$2x + 5 = 12.$$

Solution. The equation may be rewritten as

$$2x + 5 = 7 + 5.$$

Cancelling the addend 5 on the two sides, we get

$$2x = 7. \qquad \text{(using } ii\text{)}$$

Now, we cannot find any natural number x which when multiplied by 2 gives the result 7. We find, therefore, that the truth set of the equation is empty.

3 Solve the inequality

$$2x + 7 < 18.$$

Solution. The inequality can be rewritten in the form

$$2x + 7 < 11 + 7.$$

Cancelling the addend 7 on the two sides, we have

$$2x < 11.$$

By trial we see that 1, 2, 3, 4, 5 make this last statement true. Thus the solution set of the inequality consists of the numbers 1, 2, 3, 4, 5.

4. Find the truth set of

$$3x + 2 > 11 ;$$

the domain for x being given as the set of all odd natural numbers.

Solution. The given inequality can be rewritten as

$$3x + 2 > 9 + 2$$

and so $$3x > 9 \qquad i.e., \qquad 3x > 3 \times 3.$$

$$\therefore \qquad x > 3.$$

The truth set consists of 5, 7, 9, 11...

EXERCISES

1. Find the truth sets of the following equations, the domain of the variable in each case being given as the set of natural numbers.

(*i*) $x + 55 = 73$ (*ii*) $82 + x = 94$

(*iii*) $61 + x = 87$ (*iv*) $87 + y = 90$

(*v*) $y + 36 = 62$ (*vi*) $y + 3 = 60$

(*vii*) $x + 13 = 13$ (*viii*) $2x = 16$

(ix) $2x = 5$
(x) $2x + 5 = 11$
(xi) $3 + 7y = 3y + 19$
(xii) $3y + 7 = 13$
(xiii) $9y + 5 = 4y + 9$
(xiv) $6x + 7 = 2x + 35$
(xv) $30y + 11 = 3y + 25$.

2. Solve the following inequalities for x, the domain of the variable in each case being the set of natural numbers.

(i) $x + 3 > 5$
(ii) $3x > 9$
(iii) $3x + 5 > 20$
(iv) $11x + 9 > 6x + 13$
(v) $5x + 4 < 7$
(vi) $2x + 3 < 7$
(vii) $11 + 3x < 2x + 12$
(viii) $3 < 2x + 1$
(ix) $x + 1 < 7x$.

3. Solve the following open statements, the domain of the variable being the set of odd natural numbers.

(i) $87 + y = 90$
(ii) $2x = 16$
(iii) $3x + 5 < 11$
(iv) $5x + 4 < 25$
(v) $3x > 9$
(vi) $11x + 3 < 9x + 21$.

Examples

Solve the following open statements, for x, y; given that the domain of x, y is the set of natural numbers.

(i) $x + y = 11$
(ii) $x+y < 7$.

Solution. (i) Essentially $x < 11$. Now for $x=1$, y must be 10. Similarly for $x = 2$, y must be 9, and so on. Thus the truth set consists of the number pairs (1, 10), (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2) and (10, 1).

(ii) The least value that x can have is 1 and then y could be 1, 2, 3, 4, 5. Similarly, when x is 2, y can be any one of 1, 2, 3, 4 and so on. Thus the truth set consists of the number pairs (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2) and (5, 1).

EXERCISES

Solve the following open statements, given that x, y are natural numbers.

(i) $2x + y = 17$
(ii) $2x + 3y = 21$
(iii) $2x + y < 2$
(iv) $2x + 5y = 6$.

6. SUBTRACTION AND DIVISION

Subtraction. Given any two natural numbers a and b, can we *always* find a natural number d such that

$$a = b + d\,?$$

The answer to this question is '*No*' inasmuch as, if a were 3 and b were 5, we cannot find a natural number d such that

$$3 = 5 + d.$$

In fact in terms of the order relation '$>$' we know that given two natural numbers a and b, the natural number d such that

$$a = b + d$$

exists only if we have $a > b$.

The number d, in case it exists, is called the **Difference** of the numbers a and b, taken in this very order, and is denoted by

$$a - b$$

to be read as '*a minus b*,' or '*b subtracted from a*'. Thus another way of writing the equality

$$a = b + d$$

is $\quad a - b = d.$

The process of finding the number $a - b$ is called subtraction *so that subtraction is the inverse of addition.* Surely, $a - b$ is the number which when added to b gives as the resulting sum, the number a.

Through subtraction we are able to associate, to a pair of natural numbers a and b, taken in this very order, the natural number $a - b$, provided, of course, $a > b$. Because of this limitation, we cannot subtract any given natural number from any other given natural number. We say that the *set of natural numbers is not closed with respect to subtraction.* The reader should note at this stage that the set of natural numbers is closed *w.r.t.* (with reference to) addition inasmuch as we can always add any two natural numbers to obtain their sum as a natural number.

We may as well note here that whenever it is possible to subtract b from a, it shall not be possible to subtract a from b Therefore, the question of the commutative property of subtraction in the set of natural numbers does not arise.

Again, let us consider three natural numbers, say, 25, 12, 3 taken in this very order. By the two different manners of association, we have the natural numbers

$$(25 - 12) - 3 \qquad \text{and} \qquad 25 - (12 - 3).$$

These two numbers turn out to be 10 and 16 respectively. What could be our conclusion now ? In fact we see that subtraction in the set of natural numbers is **not** associative.

Note. On account of the limitation of our not being always able to subtract one natural number from another, we should note that whenever we come across the symbol

$$a - b$$

it has to be understood that we are working under the condition

$$a > b,$$

even if the condition is not explicitly mentioned.

Examples.

1. Show that

$$a(b - c) = ab - ac$$

provided $b > c$, whatever natural numbers a, b, c may be.

Solution

$$b > c$$
$$\Rightarrow ab > ac$$
$$\Rightarrow ab - ac \quad \text{is meaningful.}$$

Now $a(b-c) + ac = a[(b-c)+c]$ *D*

$= ab,$ *Definition*

This implies

$$a(b-c) = ab - ac.$$

2. For what values of x is the expression

$$(2x - 8) - x$$

meaningful, x being a natural number ?

Solution. $2x - 8$ is meaningful provided

$$2x > 8,$$

which is the same as $x > 4.$

Again $(2x - 8) - x$ is meaningful provided

$$(2x - 8) > x.$$

Adding 8 to both sides we get its equivalent form

$$2x > x + 8$$

or $x + x > x + 8.$

Cancelling the addend x on both sides, we get

$$x > 8.$$

So, the given expression is meaningful provided

$$x > 4 \text{ and } x > 8.$$

Thus, if and only if $x > 8$, the given expression is meaningful. Thus, x must b one of the numbers 9, 10, 11, 12...

3. Solve the equation

$$8 - 5x = 3,$$

the domain of the variable being the set of natural numbers.

Solution. We have that x must be such that

$$8 > 5x$$

and this is possible only if $x = 1.$

Again $8 - 5x = 3$

is equivalent to $8 = 5x + 3.$ (Definition of subtraction)

This is again equivalent to

$$5 + 3 = 5x + 3$$

which is equivalent to $5 = 5x$

which is equivalent to $1 = x.$

This, 1 is the only solution of given equation.

Check. Writing for x, 1, we have

$$8 - 5 . 1 = 3$$

as a true statement.

EXERCISES

1. Wherever possible subtract the second number from the first.

(*i*) 25, 11 (*ii*) 37, 39

(*iii*) 14, 12 (*iv*) 15, 15.

2. For what values of x are the following expressions meaningful, x being a natural number ?

(*i*) $4 - x$ (*ii*) $x - 9$ (*iii*) $x - (3x - 8)$.

3. Solve the following open statements, the domain of the variable being the set of natural numbers.

(*i*) $x - 14 = 90$ (*ii*) $7 - 3x = 4$

(*iii*) $x - 3 > 7$ (*iv*) $2 - 3x < 1$

(*v*) $3 - x < 8$ (*vi*) $3x - 2 < 5$.

Division

Analogous to subtraction as the inverse of addition, we have *division as the inverse of multiplication*.

Given any two natural numbers a and b, is it always possible to find a natural number c snch lhat

$$a = bc\ ?$$

We examine this question in the specific case when

$$a = 3 \text{ and } b = 2.$$

Can we find a natural number c such that

$$3 = 2c\ ?$$

The answer to this question is *No*. In fact, we have

$$2 \times 1 = 2 \quad \text{and} \quad 2 \times 2 = 4.$$

Also if $\quad c > 2 \quad \text{then} \quad 2c > 2 \times 2$

i.e., $\quad 3c > 4$

and so we cannot find any natural number c such that

$$3 = 2c.$$

However, depending upon the choice of the numbers a and b, we may be able to find c such that

$$a = bc.$$

As for example if $\quad a = 12 \quad \text{and } b = 3 \quad \text{then} \quad c = 4$

because $\quad 12 = 3 \times 4.$

Definition. *Given two natural numbers a and b, if there exists a natural num. c such that* $\quad a = bc$

then we write $\quad c = a \div b$

to be read as $\quad$ '*a divided by b' equals c*

and say that *a is divisible by b*. We also say *that b is a factor of a* or equivalently *a is a multiple of b*.

The process of obtaining the number $a \div b$ is called *Division*. Surely $a \div b$ is a natural number, if it exists, such that when multiplied by b gives as the product the natural number a.

Through division, we are able to associate, *in the limited sense* to a pair of numbers, a, in this very order, a natural number

$$a \div b$$

in case it exists. For two distinct numbers a and b, whenever b is a factor of a, a will not be a factor of b. But if a and b were equal, then the statements a is a factor of b and b is a factor of a, will be simultaneously true. For example, if

$$a = 13 = b$$

then a is a factor of b and b is a factor of a, because there exists the natural number 1 such that

$$13 = 1 \,.\, 13.$$

Because of the limitation of our not being always able to divide a natural number by another, we say that *the set of natural numbers is not closed for division* whereas *it is closed for multiplication.*

We note that whenever it is possible to divide a by b, we cannot divide b by a, unless of course, both a and b are the same. So that division in the set of natural numbers is not commutative

Again, let us consider three numbers, say, 72, 12, 2, taken in this order. Then

$$(72 \div 12) \div 2 \qquad \text{and} \qquad 72 \div (12 \div 2)$$

are the two numbers obtained from the three given numbers through division by two different manners of association. These two numbers come out to be

$$6 \div 2 \qquad \text{and} \qquad 72 \div 6$$

i.e. 3 and 12 respectively.

We, therefore, see that division in the set of natural numbers does not possess the *Associative Property*.

Note. In view of the limitation of our not being always in a position to divide a natural number by another, whenever, we come across the symbol

$$a \div b$$

we have to bear in mind that we are working under the condition that 'a is a multiple of b' or equivalently 'b is a factor of a', even though it may not be explicitly mentioned.

Even though it is true that one has a sense of dissatisfaction in respect of the limitations of subtraction and division, yet it is an interesting fact that the limitation with regard to division in the set of natural numbers has led to a very extensive and beautiful theory known as **Theory of Numbers.** We shall be considering the elementary aspects of this Theory of Numbers in the following chapter 2.

It may not be out of place to mention here that several Indian mathematicians specially *Ramanujam* have had the credit of making original contributions to this theory.

EXERCISES

1. Wherever possible, divide the first number by the second.

(*i*) 26, 13 (*ii*) 25, 15
(*iii*) 37, 37 (*iv*) 4, 12.

2. For what values of x are the following expressions meaningful, x being a natural number.

(*i*) $x \div 3$ (*ii*) $3 \div x$
(*iii*) $(x + 3) \div 2$ (*iv*) $36 \div (x + 7)$
(*v*) $36 \div (7 - x)$ (*vi*) $(x - 2) \div 6$.

3. Slove the following open statements, the domain of the variable being the set of natural numbers.

(*i*) $x \div 2 = 7$ (*ii*) $(x \div 2) + 3 = 4$
(*iii*) $[(3x) \div 4] - 3 = 9$ (*iv*) $x \div 3 > 4$
(*v*) $4 \div x < 7$ (*vi*) $4 \div x > 3$.

7. THE ORDER OF OPERATORS, GROUPING SYMBOLS, BRACKETS

We have learnt in the present chapter the properties of the operations of addition and multiplication and the inverse operations of subtraction and division. Also we have pointed out the limitations with regard to the operations of subtraction and division. Each of these operations is **Binary** in the sense that each operates on a pair of natural numbers to give as a result a natural number. The symbols used for these four operations are

$$+, -, \times, \div$$

and each of these symbols comes in between two numbers to give ultimately another number. For example

$$3 + 2 = 5,\ 5 - 4 = 1,\ 2 \times 3 = 6,\ 6 \div 2 = 3.$$

In each case we begin with two numbers and one of the four operations.

But we do come across expressions which involve more than one operation, and more than two numbers. In such a case, we have to note that the order in which these operations are performed, in general, prove material in the sense that the final result obtained after carrying out all the operations depends upon the order in which the operations are carried out. In the examples which follow we shall try to exhibit this dependence of the final result on the order of carrying out the operations.

I. Consider the expression

$$12 - 4 - 3.$$

This expression involves three numbers 12, 4 and 3 and the operation of subtraction twice. The subtraction signs are sandwiched between the numbers 4 and 3 and between 12 and 4. We can proceed in two different ways depending upon which of the two subtractions is to be first carried out.

Thus we may have

$$12 - (4 - 3)$$

or

$$(12 - 4) - 3.$$

Thus, if we execute the subtraction on the right first, and then that on the left, we have

$$12 - (4 - 3) = 12 - 1 = 11.$$

On the other hand, if we execute the subtraction on the left first and then the one on the right, we shall have

$$(12 - 4) - 3 = 8 - 3 = 5.$$

We arrive at two different results in the two cases.
So, unless along with the expression

$$12 - 4 - 3$$

we know the *order* in which the subtractions have to be carried out we shall have to face an indefinite situation.

To indicate the order in which the operations are to be performed we employ what may be called the

Grouping Symbols

or

Brackets.

The usual types of bracket pairs employed are

$$(\ ),\quad [\ ],\quad \{\ \}$$

respectively called

Circular, Square, Curly brackets.

Thus, around the two numbers pertaining to any given operation sign, we put the two brackets of a particular type. We may, however, use the same pair of brackets any number of times in the same expression.

We go back to the expression

$$12 - 4 - 3$$

and state that the two ways of computation will be precisely specified if, as indicated already, we use the grouping signs in the following manner.

(*i*) $12 - (4 - 3)$

(*ii*) $(12 - 4) - 3.$

In (*i*) we have to carry out the subtraction on the right first, whereas in the (*ii*) the subtraction on the left is to be carried out first. The ambiguity in

$$12 - 4 - 3$$

has thus been got rid of with the help ot the grouping symbols and we have precisely

$$12 - (4 - 3) = 12 - 1 = 11$$

$$(12 - 4) - 3 = 8 - 3 = 5.$$

We have now a few more examples.

II. Consider the expression

$$24 \div 6 \div 2$$

involving two divisions. We have

$$24 \div (6 \div 2) = 24 \div 3 = 8$$
$$(24 \div 6) \div 2 = 4 \div 2 = 2.$$

III. Consider the expression

$$24 \div 6 \times 2.$$

We have

$$24 \div (6 \times 2) = 24 \div 12 = 2$$
$$(24 \div 6) \times 2 = 4 \times 2 = 8.$$

IV. Consider, $24 \times 6 \div 2.$

We have

$$24 \times (6 \div 2) = 24 \times 3 = 72$$

and $$(24 \times 6) \div 2 = 144 \div 2 = 72.$$

V. Consider, $2 + 3 \times 6.$

We have

$$2 + (3 \times 6) = 2 + 18 = 20$$
$$(2 + 3) \times 6 = 5 \times 6 = 30.$$

VI. Consider, $a + b + c.$

We have

$$a + (b + c) = (a + b) + c$$

and for the expression

$$a \times b \times c,$$

we have

$$a \times (b \times c) = (a \times b) \times c.$$

We see that the order of carrying out the operation in general is material. However, we do have cases of the type IV, or VI, where there is no ambiguity, as we have the final results the same in these cases.

Even if an expression involves more than two operations at the same time, by adopting grouping symbols, we can get rid of the ambiguity which may be there.

For example, consider the expression

$$a + b \div c \times d$$

involving four numbers and three compositions. As such, the expression is ambiguous. By using, brackets, we get numbers which are in general different but each one of them is specific, once the brackets have been put. The different expressions we get are

(*i*) $[(a + b) \div c] \times d$
(*ii*) $[a + (b \div c)] \times d$
(*iii*) $a + [b \div (c \times d)]$
(*iv*) $a + [(b \div c) \times d]$
(*v*) $(a + b) \div (c \times d).$

It may be noted that the positions of the numbers and the operation symbols have been kept the same in all the five cases.

We close this section by remarking that although you might have been told about the usual order in which the various operations are to be carried out, yet it is only a matter of convention. We will, therefore, throughout this book, use brackets to indicate the order in which the operations have to be carried out.

EXERCISES

Obtain all the different specific numbers from the following, by using brackets.

(*i*) $16 + 8 \div 2 \times 4$ (*ii*) $17 + 9 - 2 \times 4$

(*iii*) $72 \div 6 - 3 \times 1$ (*iv*) $8 \times 6 - 1 + 3$.

8. SETS, STATEMENTS, SYMBOLISM

All along, in the present chapter we have been talking of the set of natural numbers. In the present section, we shall acquaint ourselves with the language and notation of sets and the symbolism of statements.

A set is a *collection of objects* and each of the objects of the collection is called *an element of or a member of the set.* As examples of sets, we may have :

(*i*) the set of all students of your school.
(*ii*) the set of all human beings in Delhi.
(*iii*) the set of all natural numbers.
(*iv*) the set of all natural numbers less than 5.
(*v*) the set of all even natural numbers.
(*vi*) the set of all multiples of 7.
(*vii*) the set of all powers of 2.
(*viii*) the set of all natural numbers x, such that $2x + 1 = 9$.
(*ix*) the set of all natural numbers x, such that $2x + 3 < 7$.
(*x*) the set of all natural numbers x, such that $3x + 1 > 7$.
(*xi*) the set of numbers 2, 7, 11.
(*xii*) the set of factors of 12.

In order to describe a set, we list all its elements and enclose them by curly brackets. Thus, for example, the sets *iv*, *v*, *vii*, *viii*, *xi* and *xii* can be described as

(*iv*) $\{1, 2, 3, 4\}$.
(*v*) $\{2, 4, 6, 8, \ldots\ldots\}$.
(*vii*) $\{2, 2^2, 2^3, 2^4, \ldots\ldots\}$.
(*viii*) $\{4\}$.
(*xi*) $\{2, 7, 11\}$.
(*xii*) $\{1, 2, 3, 4, 6, 12\}$.

EXERCISES

Describe the sets (*iii*), (*iv*), (*ix*) and (*x*), by listing their elements.

We can have a distinction between the sets of the type (*iv*), (*xi*) on the one hand and the sets (*iii*), (*v*), (*vii*), on the other. If we were asked to list all the elements of the set (*iii*) or (*v*) or (*vii*), we shall not be able to do so. We say that the sets of this form are *infinite* while those of the type (*iv*) and (*xi*) are *finite*.

EXERCISE

(*i*) Put down five finite sets.

(*ii*) Put down three infinite sets.

Consider another example of a set consisting of natural numbers which make the statement

$$3x + 7 = 2$$

true. We see that the number 7 being greater than 2, no natural number x is possible. We have already given a specific name to such a set. If you recollect, we called such a set an **Empty set.** We say, that a set consisting of no object is called the **Empty, Void** or **Null set.** It is customary to denote such a set by the symbol ϕ, to be read as "phi".

In general, capital letters of the alphabet, such as S, T, A, B, C are used to denote the sets. The set of natural numbers

$$\{1, 2, 3, \ldots\ldots\}$$

will be symbolically referred to as N.

We have called an object of a set as its element. Now, if a is an element of a set S, we write

$$a \in S$$

to be read as '*a belongs to S*' or '*a is an element of S*' or '*a is a member of S*'.

EXERCISE

Let S be the set $\{2, 7, 11\}$. Which of the following statements are true and which are false ?

(*i*) $2 \in S$ (*ii*) $8 \in S$

(*iii*) $4 \in S$ (*iv*) $5 \in S$.

If an object a is not a number of a set S, we write

$$a \notin S$$

to be read as 'a does not belong to S'.

EXERCISE

Let S be the set $\{1, 2, 3, 4, 6, 12\}$; which of the following statements are true and which are false ?

(*i*) $2 \notin S$	(*ii*) $6 \notin S$
(*iii*) $5 \notin S$	(*iv*) $3 \in S$
(*v*) $1 \in S$	(*vi*) $12 \notin S$
(*vii*) $12 \in S$	(*viii*) $4 \notin S$

Notation. In the next chapter, the reader will see that we shall have to deal with the sets of factors of natural numbers. We adopt here a notation to denote by Fac a, the set of all factors of the number a, for example

$$Fac\ 12 = \{1, 2, 3, 4, 6, 12\}.$$

EXERCISE

Describe the following sets.

(*i*) *Fac* 13	(*ii*) *Fac* 25
(*iii*) *Fac* 36	(*iv*) *Fac* 53
(*v*) *Fac* 1	(*vi*) *Fac* 24
(*vii*) *Fac* 60	(*viii*) *Fac* 45

Set Builder Notation. Instead of listing various elements of a set, it is sometimes more convenient to use the description of the set and describe it in terms of what is commonly known as the set builder notation. For example, the set $\{2, 2^2, 2^3, 2^4, \ldots\ldots\}$ described as

$$\{2^n : n \in N\}.$$

This means, that the set consists of the natural numbers 2^n obtained by replacing n by each member of the set N, *i.e* by each natural number. The expression may by read as follows :

The set consists of the numbers 2^n such that, n belongs to N, so that ' ; ' is read as '*such that*'.

Again, the set $\{1, 2, 3, 4, 5, 6\}$ could be described as

$$\{x : x < 7, x \in N\}.$$

The set $\{4\}$ could be described as

$$\{x : 2x + 1 = 9, x \in N\}$$

and the set $\{5, 10, 15, 20, \ldots\ldots\}$ could be described as

$$\{5a : a \in N\}.$$

Equality of sets

Definition. *Two sets S and T are said to be equal if every member of S is a member of T and conversely every member of T is a member of S.*

The reader may easily see that the following are true statements.

(*i*) $\{5a : a \in N\} = \{5, 10, 15, 20, \ldots\ldots\}$

(*ii*) $\{1, 3, 5\} = \{5, 1, 3\}$

(*iii*) $\{x : 3x + 1 > 7, x \in N\} = \{3, 4, 5, 6 \ldots\ldots\}$

We should note that a set is just a collection and the order in which the elements of the set occur is immaterial. Thus for example, we have

$$\{1, 3, 5, 7\} = \{7, 5, 3, 1\}.$$

Also the repetition of any element does not alter the set. Thus, we have

$$\{7, 3, 5, 5, 1\} = \{1, 1, 7, 3, 5, 3\}.$$

Sub-set of a Set, Super-set of a Set

Definition. *A set T is said to be a sub-set of a set S, if every member of T is a member of S. In symbols we write $T \subset S$,*

to be read as '*T is a sub-set of S*,' or '*T is contained in S*'. Again if T is a sub-set S, we also say that S is a super-set of the set T. In symbols, we write

$$S \supset T$$

to be read as '*S is a super-set of T*' or 'S contains T'. For example,

(*i*) $\{1, 3, 5\} \subset \{3, 5, 7, 1\}$.

(*ii*) The set of students in your class is a sub-set of the set of students of your school.

(*iii*) The set of people living in Delhi is a sub-set of the set of people in India.

(*iv*) The set of natural numbers is a super-set of the set of factors of 36. Symbolically, we may write this as

$$\mathbf{N} \supset Fac\ 36.$$

Again, we have that $Fac\ 36 \subset Fac\ 36$

i.e., the set of factors of 36 is a sub-set of itself.

In fact, every set S is a sub-set of itself. All that we need to see is that every element of S, must be an element of S, which is certainly so. Thus, for any arbitary set S

$$S \subset S$$

is a true statement.

Also, given any set S, we shall always have

$$\phi \subset S$$

i.e., the Void set is a sub-set of every set S. In order to see this, we must note that every element of ϕ must be an element of S. In other words, we should see that there is no member of ϕ which is not a member of S. Now, because ϕ contains no element, the result follows immediately.

We conclude, therefore, that every non-empty set S has at least two sub-sets namely ϕ and S.

Definition. *A sub-set T of a set S, other than ϕ and S is called a proper sub-set of S.*

Caution. The student is warned against any possible confusion between the symbols

$$\in, \subset$$

for 'belongs to' and 'contained in' respectively. In any statement to the left of the

symbol $\in$ must there be an element and to its right a set, whereas the symbol $\subset$ must be sandwiched between two sets on either of its two sides.

Example

$$\{1\} \subset \{1, 2, 3, 4\}$$
$$1 \in \{1, 2, 3, 4\}.$$

We should note here that 1 is an element whereas {1} is a set consisting of the element 1.

EXERCISE

Put down all possible sub-sets of the following sets.

(*i*) {1, 3, 5} (*ii*) {7}
(*iii*) {2, 6} (*iv*) ϕ.

Given any two sets S and T, we may have neither as a sub-set of the other. For example, consider the set of factors of 24 and 36 respectively. The reader is advised to put down the two sets and see why none of the two is a sub-set of the other.

EXERCISE

Prove the following :

(*i*) Fac 12 $\subset$ Fac 36 (*ii*) Fac 30 $\supset$ Fac 15

(*iii*) Fac 18 is not a sub-set of Fac 30. (*iv*) Fac 30 is not a sub-set of Fac 18.

Least and Greatest Members of a Set of Numbers.

Consider the set **N** of natural numbers. Certainly, this set has a member, namely, 1, such that it is less than or equal to any member of **N**. We call this number 1 the least of the set **N**.

In fact, *every non-empty sub-set of the set* **N**, *will always have the least element.* This property of the set of natural numbers is referred to by saying that the set **N** *of natural numbers is well ordered.*

If we consider any finite sub-set of **N**, say

$$S = \{1, 3, 6, 2, 18\},$$

it has a member 18 which is greater than or equal to every member of S. This number, 18 is called the greatest of all the members of S. The set **N** does not have the greatest member.

Also a finite sub-set of **N** has the greatest member whereas an infinite sub-set of **N** does not have the greatest member.

EXERCISE

Put down the least and the greatest (if it exists) members of the following :

(*i*) $\{2, 4, 6, 8, \ldots\}$ (*ii*) $\{2, 4, 16, 256\}$

(*iii*) $\{10, 11, 12, 13, \ldots\}$ (*iv*) $\{10, 15, 35, 70\}$

(*v*) $\{1\}$.

Operations on Sets. Union and Intersection.

Consider any two sets S and T. The set consisting of objects which are either elements of S or elements of T or of both is called the union of the sets S and T and is symbolically written as $S \cup T$, where $\cup$ is the symbol for union.

Thus, $S \cup T = \{x : x \in S \text{ or } x \in T \text{ or } (x \in S \text{ and } x \in T)\}$

For example if $S = \{1, 3, 5, 7 \ldots\}$

and $T = \{2, 4, 6, 8, \ldots\}$,

then $S \cup T = \mathrm{N}$, the set of natural numbers.

Again, if $S = \{1, 2, 3, 6\}$

and $T = \{1, 2, 4\}$,

then $S \cup T = \{1, 2, 3, 4, 6\}$.

The reader should note that we do not write an element again once it has been written in that it may be seen that 2 which is a member of S as also of T has been taken only once.

EXERCISES

1. Compute $S \cup T$ for the following sets.

(*i*) $S = \{1, 2, 3\}$, $T = \{1, 2, 3, 4\}$

(*ii*) $S = \{1, 3, 7\}$, $T = \{4, 6, 9\}$

(*iii*) $S = \{1, 3, 5, 15\}$, $T = \{1, 2, 3, 4, 6, 12\}$.

2. Compute $T \cup S$ for all the three cases of Ex. 1, and see that
$$S \cup T = T \cup S.$$

3. Compute $(A \cup B) \cup C$ and $A \cup (B \cup C)$ for the three sets.
$$A = \{2, 4, 6\}$$
$$B = \{1, 7, 6\}$$
$$C = \{1, 8, 9\}$$
and verify that $(A \cup B) \cup C = A \cup (B \cup C)$.

4. Name the properties of union of sets you have verified in Ex. 2 and 3.

5. Put down any three sets A, B and C and Compute $(A \cup B) \cup C$ and $(C \cup A) \cup B$.

Intersection. Consider any two sets S and T. The set consisting of those elements which are members of S and as well of T is called the *Intersection* of the two sets S and T and is symbolically written as
$$S \cap T.$$

Thus, $S \cap T = \{x : x \in S \text{ and } x \in T\}$.

For example, if $S = \text{Fac } 36$ and $T = \text{Fac } 24$

then $S = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

and $T = \{1, 2, 3, 4, 6, 8, 12, 24\}$

$\therefore$ $S \cap T = \{1, 2, 3, 4, 6, 12\}$.

Again, if $S = \{1, 3, 5\}$, $T = \{1, 5\}$,

then $S \cap T = \{1, 5\}$.

Also, if $S = \{1, 3, 5\}$ and $T = \{4, 6, 7\}$,

then $S \cap T = \phi$,

as there is no common member of S and T.

EXERCISES

1. Campute $S \cap T$ and $T \cap S$ for sets S and T given in Ex. 1 of the previous section, and see that $S \cap T = T \cap S$.

2. Compute $(A \cap B) \cap C$ and $A \cap (B \cap C)$ for the sets A, B, C of Ex 3 of the last section.

3. Put down any three sets A, B, C and compute the following.

(*i*) $A \cap (B \cup C)$	(*ii*) $(A \cap B) \cup (A \cap C)$
(*iii*) $A \cup (B \cap C)$	(*iv*) $(A \cup B) \cap (A \cup C)$
(*v*) $A \cup \phi$	(*vi*) $A \cap \phi$.

Statements. We have been dealing with statements which are true, false or open. In the following, we introduce symbols, which make it convenient to handle statements and their interrelations.

Suppose we have two statements P and Q. Then we have quite often a situation where we say

'*If* P *then* Q',

as for example,

'*If* $a > b$ *then* $a + c > b + c$.'

However, we assume that a, b, c are all natural numbers.

In symbols, we write

$$P \Rightarrow Q$$

to be read as P *implies* Q.

Thus, for example, if x is any natural number, we have

$$x > 3 \Rightarrow x^2 > 3^2.$$

Again, we may have two statements P and Q such that P *is implied by* Q. In symbols, we write

$$P \Leftarrow Q.$$

For example, if a, b, c are any natural numbers, then the cancellation law for multiplication gives,

$$a = b \Leftarrow ac = bc.$$

Quite often, we have statements P and Q which are such that

$$P \Rightarrow Q$$

and
$$P \Leftarrow Q.$$

We write, in such a case

$$P \Leftrightarrow Q$$

to be read as 'P implies Q and is implied by Q'. We also say, then, the statements P and Q are equivalent. As for example,

$$a = b \Rightarrow ac = bc$$

and
$$a = b \Leftarrow ac = bc$$

for all natural numbers a, b, c. The first of these follows from the definition of multiplication. We combine the two and write

$$a = b \Leftrightarrow ac = bc$$

so that the statement, $a = b$ and $ac = bc$ are equivalent, a, b, c being any natural numbers

EXERCISE

Show that

(i) $$2x + 3 > 7 \Leftrightarrow x > 2$$

(ii) $$x + 5 = 2x + 1 \Leftrightarrow x = 4.$$

We have also, throughout the chapter come across three phrases 'For all', 'There exists' and 'Such that'. Even in the later chapters, the reader will find very frequent occurence of these phrases. We adopt the following symbols for the same.

"For all"	$\forall$
"There exists"	$\exists$
"Such that"	:

In terms of the symbolism of this section, we shall try to restate the basic results of this chapter in a neat and compact form. It may be seen that a good deal of *economy* in terms of space, time and effort is effected, once the student becomes familiar with the use of the symbols.

For example, we know that in natural numbers $a > b$ if there exixts a natural number d such that $a = b + d$.

Also, if there exists a natural number d such that

$$a = b + d$$

then
$$a > b.$$

In terms of the symbols, this entire statement can be written in the form

$$a > b \Leftrightarrow \exists\ d \in \mathrm{N} : a = b + d \ \forall\ a, b \in \mathrm{N}$$

i.e., for all natural numbers a and b, a is greater than b implies and is implied by the existence of natural number d such that a equals the sum of b and d.

9. DIVISION ALGORITHM

We should note that, in the set N of natural numbers, there exists the possibility that neither of the two given natural numbers is a factor of the other. As for example, of the two natural numbers 4 and 6, neither 4 is a factor of 6 nor 6 is a factor of 4.

This means that if a, b are given natural numbers, the natural number a may not be divisible by b. The reader may recall that in such situations he has been, in his primary and middle classes, talking of the quotient and the remainder obtained on divinding a by b.

In the following we attempt to present this procedure in a little more formal manner.

Let us consider two numbers, say, 9 and 24. We have $24 > 9$. Also the set ot multiples of 9 consists of the numbers

$$1 \times 9, 2 \times 9, 3 \times 9, 4 \times 9, \ldots\ldots\ldots$$

and we see the number 24 is not a member of this set *i.e.*, 24 is not a multiple of 9. We observe that, to begin with, the multiples are less than 24 but after a certain stage we come across a multiple which is just greater than 24. The multiple of 9 just preceding this is the greatest of those multiples which are less than 24. These two multiples are 3×9 and 2×9 respectively. In fact we see that the number 24 is sandwiched between the two consecutive multiples of 9 *viz.*, 2×9 and 3×9, so that, we have

$$2 \times 9 < 24 < 3 \times 9.$$

Also, because $2 \times 9 < 24$

there exists a natural number, namely 6 such that

$$24 = 9 \times 2 + 6.$$

Thus, there exists two numbers 2 and 6 respectively called the *quotient* and the *remainder*, such that

$$24 = 9 \times 2 + 6.$$

We may note that the uumber 6 (the remainder) is essentially less than the number 9 (the divisor).

Again, let us start with two numbers say 20 and 3. The set of multiples of the number 3 consists of the numbers

$$1 \times 3, 2 \times 3, 3 \times 3, 4 \times 3, 5 \times 3, 6 \times 3, 7 \times 3, 8 \times 3, \ldots .$$

and 20 is not a member of this set. However,

$$6 \times 3 < 20 < 7 \times 3$$

and $$20 = 3 \times 6 + 2.$$

The quotient in this case is 6, the remainder 2 which is less than the divisor 3.

EXERCISE

Repeat the process of the two illustrations above for the following pairs of numbers.

(*i*) 12 and 35 (*ii*) 19 and 215

(*iii*) 104 and 1387 (*iv*) 224 and 1345.

The discussion above leads us to the following result called *Division Algorithm.*

Theorem. Given two natural numbers a and b, $b \neq 1$, $a > b$ and a is not divisible by b, there exist two numbers (unique) q and r such that

$$a = bq + r \qquad r < b.$$

Proof. The set of multiples of b, consists of the numbers

$$b, 2b, 3b, 4b, \ldots\ldots$$

As b is not a factor of a, we see that a is not a member of the above set.

In the beginning, the multiples of b are less than a. But after a certain stage we shall get a multiple of b such that it is just greater than the number a.

Let bq denote the greatest multiple of b such that

$$q\,b < a$$

and

$$(q+1)\,b > a.$$

Thus we have

$$q\,b < a < (q+1)\,b.$$

Now, because $q\,b < a$ there exists a natural number r, say, such that

$$a = b\,q + r.$$

Also because

$$a < (q+1)\,b$$

we have

$$b\,q + r < bq + b$$

i e.,

$$r < b.$$

We have two numbers q and r such that

$$a = bq + r \quad \text{and} \quad r < b.$$

Also, the manner of obtaining the numbers $q\,b$ and $(q+1)\,b$ shows that q is unique. Also since q is unique it follows that r is also unique. Hence the theorem.

EXERCISE

Determine the quotient and the remainder for the following pairs of natural numbers.

(*i*) 17, 3 (*ii*) 286, 11

(*iii*) 575, 17 (*iv*) 9087, 22

(*v*) 5555, 25 (*vi*) 7707, 77

SUMMARY OF THE BASIC RESULTS

	$(a+b)=(b+a)$	$\forall\ a, \in \mathrm{N}$	*CA*
II.	$(a+b)+c=a+(b+c)$	$\forall\ a, b, c \in \mathrm{N}$	*A A*
III.	$a \cdot b = b \cdot a$	$\forall\ a, b \in \mathrm{N}$	*CM*
IV.	$(ab)\,c = a\,(bc)$	$\forall\ a, b, c \in \mathrm{N}$	*AM*
V.	$a \times 1 = a$	$\forall\ a \in \mathrm{N}$	*MO*
VI.	$a\,(b+c) = ab + ac$	$\forall\ a, b, c \in \mathrm{N}$	

Trichotomy Law

VII. For any natural numbers a, b one and only one of the following possibilities is there.

(1) $a = b$ (2) $a > b$ (3) $b > a$

Transitivity Law

VIII.	$a > b$ and $b > c \Rightarrow a > c$	$\forall\ a, b, c \in \mathrm{N}$
IX.	$a > b \Leftrightarrow a + c > b + c$	$\forall\ a, b, c \in \mathrm{N}$
X.	$a > b \Leftrightarrow ac > bc$	$\forall\ a, b, c \in \mathrm{N}$
XI.	$a = b \Leftrightarrow a + c = b + c$	$\forall\ a, b, c \in \mathrm{N}$
XII.	$a = b \Leftrightarrow ac = bc$	$\forall\ a, b, c \in \mathrm{N}$

Well-ordering Property

XIII. Any non-empty sub-set S of N has the least number in it.

Division Algorithm

XIV. Given any two natural numbers a and b, $b \neq 1$, $a > b$ and a is not divisible by b, There exist two numbers (unique) q and r such that

$$a = bq + r \qquad r < b.$$

10. PROBLEMS

In this section, we will make use of the properties of the set of natural numbers to solve various types of problems. We shall see that to solve a problem, we have first to translate it into an equation or an inequality, whose solution will ultimately give us a solution of the given problem.

Before giving some examples we give illustrations of translating sentences in ordinary language in a mathematical form and *vice versa.* It will be of interest to note here that whereas a sentence in ordinary language will give rise to same mathematical form, a mathematical form can be interpreted in more than one way.

Examples

1. *'The sum of two consecutive natural numbers is 67'*, can be written in mathematical language in the form

$$x + (x + 1) = 67.$$

2. The mathematical form

$$x + (x + 1) = 67$$

may be interpreted in many different ways, as for example

(*i*) the sum of two consecutive numbers is 67.

(*ii*) The sum of the ages of two brothers, one of whom is one year older than the other, is 67.

(*iii*) The total earnings of an individual on two days is Rs. 67 and he earns one rupee more on the second day than on the first.

(*iv*) The total distance travelled by a car in two hours is 67 kilometres, if the distance travelled in the second hour is one kilometre more than that in the first hour.

The variable x, represents

(*i*) the first number.

(*ii*) the age of the younger brother.

(*iii*) the individual's earnings on the first day.

(*iv*) the distance travelled by the car during the first hour.

The student may appreciate that the same equation has been seen to correspond to four different situations. Of course, we could have many more such interpretations.

EXERCISES

Translate the following sentences in ordinary language into mathematical language.

(*i*) The product of two consecutive numbers is 18.

(*ii*) The sum of two numbers is 57 and the greater number is 13 more than the smaller one.

(*iii*) The number of boys in a class is 13 more than that of the girls and the total number of students is 57.

(*iv*) The age of the father is three years more than twice the age of the son and the total of their ages is 63 years.

(*v*) The perimeter of a rectangle is 42 cm. and its length is 3 cm. more than the width.

2. In each case of Ex. 1 above, what does the variable stand for ?

3. Interpret each of the followidg open statements in at least two different ways.

(*i*) $x + 5 = 43$ (*ii*) $x + (2x + 1) = 52$

(*iii*) $x + (x + 1) < 32$ (*iv*) $2x + 2(x + 1) = 42$

(*v*) $x - 5 > 3x - 13$.

Problem 1. The sum of two consecutive multiples of 3 is 171. Find the numbers.

Solution. The problem states that the sum of two consecutive multiples of 3 is 171, so that we have to have expressions for finding consecutive multiples of 3. We start with the assumption that one of them is $3x$. The other one will be then $3(x+1)$. Also, the sum of the two is given to be 171 so that, we have

$$3x + 3(x+1) = 171$$
$$\Leftrightarrow 6x + 3 = 171$$
$$\Leftrightarrow 6x + 3 = 171 + 3$$
$$\Leftrightarrow \quad 6x = 168$$
$$\Leftrightarrow \quad 6x = 6 \times 28$$
$$\Leftrightarrow \quad x = 28.$$

Also then $\qquad 3x = 84$ and $3(x+1) = 87.$

The two required numbers, therefore, are 84, 87.

Problem 2. In three hours a car travels a total of 153 kilometres. Find the distance travelled during the first hour, if the distance covered during the second hour is twice that in the first hour, and that covered in the third hour is 5 kilometres less than that during the second hour.

Solution. Let the distance covered in the first hour be x km.

Then the distance covered during the second hour $= 2\,x$ km. and that covered during third hour $= (2x - 5)$ km.

As the total distance covered in three hours is 150, we have

$$x + 2x + (2x - 5) = 150$$
$$\Leftrightarrow x + 2x + (2x - 5 = 150 + 5$$
$$\Leftrightarrow x + 2x \times 2x \quad = 155$$
$$\Leftrightarrow \quad 5x \quad = 155$$
$$\Leftrightarrow \quad x \quad = 31.$$

Thus the distance covered during the first hour is 31 km.

Problem 3. Distribute Rs. 500/- amongst Anita, Kavita and Anupa in such a way that Kavita gets Rs. 20 less than the double of what Anita gets and Anupa gets Rs. 50 more than what Kavita gets.

Solution. Let Anita get Rs. x.

Then Kavita gets Rs. $(2x - 20)$

and Anupa gets Rs. $[(2x - 20) + 50]$.

The given problem corresponds to the open statement

$$x + (2x - 20) + (2x + 30) \quad = 500$$
$$\Leftrightarrow x + (2x + 30) + (2x - 20) = 500$$
$$\Leftrightarrow x + (2x + 30) + 2x \quad = 500 + 20 = 520$$
$$\Leftrightarrow \quad 5x + 30 \quad = 520 = 490 + 30$$
$$\Leftrightarrow \quad 5x \quad = 490$$
$$\Leftrightarrow \quad x \quad = 98$$

Thus, Anita gets Rs. 98, Kavita gets Rs. $(98 \times 2 - 20)$, *i.e.* Rs. 166 and Anupa gets Rs. $(176 + 50)$, *i.e.* Rs. 226.

Problem 4. The perimeter of a square in centimetres is the same as its area in square centimetres. Find the side of the square.

Solution. Let the side of the square be x cm.

Then the perimeter is $4x$ cm.

Also the area of the square $= x \,.\, x$ sq. cm.

$$\therefore \quad 4x = x \,.\, x$$
$$\Leftrightarrow 4 = x.$$

Thus, the side of the square is 4 cm.

Problem 5. A rectangle has its sides measurable in full centimetres. Its length is twice its breadth and the area is given to be less than or equal to 46 sq. cm. Find all the possible values of its length.

Solution. Let x cm. be the breadth of the rectangle. Then its length is $2x$ cm.

The area of the rectangle, therefore, is $x.\,(2x)$, *i.e.* $2x^2$ sq. cm.

As this area is given to be less than or equal to 46 sq. cm., we have that

$$2x^2 \leqslant 46$$
$$\Leftrightarrow 2x^2 \leqslant 2 \times 23$$
$$\Leftrightarrow \quad x^2 \leqslant 23.$$

The set of values of x satisfying the above relation is obviously

$$\{ 1, 2, 3, 4 \}.$$

The length of the given rectangle in cm. can be 2, 4, 6 or 8.

Verification. The student is advised to verify in each case the correctness of his result.

EXERCISES

1. Find the consecutive numbers whose sum is 57.
2. Find two consecutive even numbers whose sum is 144.
3. Find two consecutive odd numbers whose sum is 68.
4. The sum of two consecutive multiples of 8 is 168. Find the numbers.
5. Find three consecutive numbers whose sum is 81.
6. Find three consecutive even numbers whose sum is 108.
7. Find three consecutive odd numbers whose sum is 327.
8. The larger of two numbers is 17 more than twice the smaller. If their sum is 104, find the two numbers.
9. Seven times a number is 12 less than thirteen times the number. Find the number.

10. Thirteen added to twice a number is the same as 5 added to 3 times the same number. Find the number.

11. The larger of two numbers is 13 more than the smaller. Their sum has to be less than 27. Find the set of all possible values of the smaller number.

12. 9 added to three times a number is greater than the excess of of five times that number over 7. Find the set of all possible values of the given number.

13. Can twice a number increased by 3 be greater than seven times the same number ?

14. Can three times a number increased by 6 be greater than ntne times the same number ?

15. The total distance covered by a car in two hours is 100 km. If the distance covered during the second hour is 20 km. less than thrice that covered during the first hour, find the distance covered during the first hour.

16. The perimeter of a square is less than 28 cm. If a side is measurable in full cm. find all possible lengths a side of the square can have.

17. A square has its side measurable in full centimetres. If the perimeter is less than 24 cm. and greater than 12 cm., find all possible lengths of the side of the square.

18. The perimeter of a rectangle is 14 cm. Find the breadth of the rectangle if the length is 2 cm. less than twice the breadth.

19. The perimeter of a triangle is 36 cm. If one of its sides is shorter than another by 5 cm , and the third side is greater than the second by 8 cm. find the three sides of the triangle.

20. Three consecutive numbers give the degree measures of the three angles of a triangle. What are these numbers ?

21. The capacity of a drum is 10 litres less than that of another. Together they can hold 92 litres. Find the capacities of the two drums.

22. Ram and Shyam have 50 rupees with them. If Ram has 4 rupees less than twice of what Shyam has, find the number of rupees with Shyam.

23. Distribute Rs. 300 amongst Ram, Shyam and Krishan in such a way that Ram gets Rs. 20 more than Shyam who gets twice of what Krishan gets.

24. In a class the total number of students is 50. It is given that the number of girls is 5 less than four times the number of boys. Find the number of boys and girls in the class.

25. In a class of 45, the number of boys is greater by 5 than three times the number of girls. Find the number of boys and girls in the class.

26. The sum of the ages of the father and the son is 65 years. Five years hence the age of the father will be double that of the son. Find their ages now
27. The sum of the ages of the father and the son is 60 years. Three years ago the age of the father was double that of the son. Find their ages now.
28. In y number, the tens digit is one less than twice the units digit. If the sum of the digits is 8, find the number.
29. The digit in the ten's place is twice the digit in the unit's place of a number consisting of two digits. This number exceeds that obtained on reversing the digits by 18. Find the number.

REVIEW EXERCISES

1. Given

$$A = \{2, 7, 3, 9, 8, 13\}$$
$$B = \{5, 7, 8, 17, 9\}$$
$$C = \{6, 4, 12, 14, 16\}.$$

What are the sets

$$A \cup B,\ B \cup C,\ C \cup A$$
$$A \cap B,\ B \cap C,\ C \cap A$$
$$A \cup (B \cap C),\ A \cap (B \cup C)$$
$$(A \cup B) \cap (A \cup C),\ (A \cap B) \cup (A \cup C).$$

2. Given

$$L = \{x : x \text{ is a factor of } 45\}$$
$$M = \{x : x \text{ is a factor of } 63\}$$
$$N = \{x : x \text{ is a factor of } 120\}$$
$$P = \{x : x \text{ is a factor of } 270\}.$$

What are the sets

$$L \cap M,\ M \cap P,\ L \cap (M \cap N)$$
$$(L \cap M) \cap (N \cup P).$$

3. Given

$$P = \{x : x \text{ is a multiple of } 6\}$$
$$Q = \{x : x \text{ is a multiple of } 8\}.$$

What is the set

$$P \cap Q.$$

4. For what values of x are the following expressions meaningful, the domain of x being **N**.

(*i*) $4 - x$ (*ii*) $x - (2x - 8)$

(*iii*) $(3x - 7) - x$.

5. For what values of x are the following expressions meaningful, the domain of x being N.

   (*i*) $(20 \div x) - x$ (*ii*) $(x \div 3) \div 5$
   (*iii*) $[(x - 7) \div 3] \div 2$.

6. Prove that
$$a^2 + b^2 \geqslant 2ab \ \forall \ a, b \in \mathbf{N}.$$

7. Show that
$$a > b \text{ and } c > d \Rightarrow ac + bd > ad + bc, a, b, c, d, \in \mathbf{N}$$

8. Is 2^{10} greater than or smaller than 1000 ?

9. Given that
$$x + y = 101 \quad \text{and} \quad x - y < 2$$
show that $x = 51$.

10. Given that
$$4x + 3y = 63 \quad \text{and} \quad x < 5$$
prove that $y = 14$.

11. Give all possible pairs of natural numbers x, y such that
$$x - y < 6 \quad \text{and} \quad x + y < 14.$$

12. If $x \geqslant y + 4$ and $y = z - 3$,
then $x \geqslant z + 1$.

13. Show that
$$(ab) \div c = a(b \div c).$$
Also state the conditions under which each side represents a meaningful expression.

14. Show that there exist natural numbers, a, b, c such that
$$(a^b)^c \neq a^{(bc)}.$$

15. Solve the following for x, given that x is a natural number.

   (*i*) $x + 56 = 78$ (*ii*) $42 + x = 93$
   (*iii*) $25 - x = 12$ (*iv*) $3x + 7 = 16$
   (*v*) $x - 7 = 7$ (*vi*) $7 - 4x = 13$
   (*vii*) $9 - 5x = 3$ (*viii*) $(x \div 2) + 5 = 8$
   (*ix*) $(x \div 3) + 7 = 6$ (*x*) $4x - 5 = 8$
   (*xi*) $13 - 7x = 2$ (*xii*) $x^2 = 9$
   (*xiii*) $x^2 + 5 = 20$ (*xiv*) $x^2 - 7 = 18$
   (*xv*) $25 - x^2 = 21$.

16. Solve the following inequalities for x, given that x belongs to the set of natural numbers.

   (*i*) $x - 8 > 3$ (*ii*) $x - 3 < 11$
   (*iii*) $x + 11 < 11$ (*iv*) $x + 22 < 23$
   (*v*) $27 + x < 39$ (*vi*) $2x + 5 < 19$
   (*vii*) $7x - 3 < 24$ (*viii*) $10x - 8 > 12$

(*ix*) $3x - 5 > 12$ (*x*) $3 + 8x \geqslant 11$
(*xi*) $27 + 5x \geqslant 37$ (*xii*) $13 - 4x \geqslant 5$
(*xiii*) $x \div 7 \geqslant 10$ (*xiv*) $x \div 3 \leqslant 7$
(*xv*) $(x \div 3) + 7 \leqslant 12$ (*xvi*) $3 \div x \leqslant 2$
(*xvii*) $(3 \div x) + 7 \leqslant 13$ (*xviii*) $x^2 \geqslant 64$
(*xix*) $25 - x^2 \geqslant 15$ (*xx*) $x^2 - 11 \geqslant 4$
(*xxi*) $x^2 + 5 \leqslant 5$. (*xxii*) $2x + 5 \neq 5x + 7$
(*xxiii*) $3x + 11 \neq 4x + 4$ (*xxiv*) $5x + 13 \ngtr 6x + 7$
(*xxv*) $4x + 5 \ngtr 3x + 11$ (*xxvi*) $5x + 3 \nless 8x + 2$
(*xxvii*) $11x + 1 \nless 7x + 25$ (*xxviii*) $3x + 7 \nless 5x + 5$

17. Solve the following for x, y, given that x and y are members of the set of natural numbers.

(*i*) $x + y = 7$ (*ii*) $7x + 3y = 15$
(*iii*) $x - 3y = 4$ (*iv*) $x - 3y = 3$
(*v*) $x^2 + y^2 = 1$ (*vi*) $x + 3 \leqslant y$
(*vii*) $y - 3 \geqslant x$ (*viii*) $2x + 3y \leqslant 25$
(*ix*) $x^2 + y^2 \leqslant 12$ (*x*) $x^2 + 3y^2 = 10$.

18. Find two consecutive multiples of seven whose sum is 329.

19. The perimeter of a rectangle is 56 cm. If its length is 4 cm. more than twice its breadth, find the length and the breadth of the rectangle.

20. Distribute Rs. 200 amongst Mohan, Sohan, and Om in such way that Mohan gets Rs. 10 more than what Sohan gets and Om gets Rs. 20 less than twice of what Sohan gets.

21. The difference of the ages of the father and the son is 25 years. 10 years hence the age of the father becomes double that of the son. Find the present age of the father.

22. The ten's digit is greater than the unit's digit of a number of two digits by 4 and the sum of the digits is 14. Find the number.

23. The area of a square whose each side is of a length measurable in full metres is greater than 10 m^2 and less than 100 m^2. Find all possible values of the length of the sides.

24. Describe all possible ways of giving toffees to Sita and Krishna out of a packet of 20 toffees such that Sita gets two toffees less than double of what Krishna gets.

25. A number consists of three digits such that the digit in the ten's place is twice and that in the hundred's place three times that at the unit's place. The number obtained on interchanging the digits in the unit's and the hundred's place is less than the given number by 594. Find the number.

2

Elementary Number Theory Divisibility in N

11. INTRODUCTION

We have seen that if a and b are any two given natural numbers, there may or may not exist a natural number c such that

$$a=bc.$$

Further, if for given natural numbers a, b, there does exist a natural number c such that $a = bc$, we write

$$a \div b = c$$

and say that a is *divisible* by b and that c is the quotient obtained on dividing a by b.

It will, thus, be seen that the symbol

$$a \div b$$

in respect of the set of natural numbers is meaningful if and only if a is divisible by b.

Thus, while each of the expressions

$$6 \div 2,\ 16 \div 4,\ 18 \div 3$$

is meaningful, none of the expressions

$$6 \div 4,\ 16 \div 5,\ 3 \div 6$$

has any meaning in respect of the set of natural numbers.

12. DIVISIBILITY RELATION

If a is divisible by b, we write

$$b \mid a$$

the line between b and a being vertical and not inclined.

The symbol

$$b \mid a$$

is to be read as

a is divisible by b.

For example, since 6 is divisible by 3, we write

$$3 \mid 6.$$

Again, because 30 is divisible by 5, we write

$$5 \mid 30.$$

There are several different and alternative forms in which the symbol

$$b \mid a$$

could as well be read and we give these in the following. But before we do this, we refer to the concepts of :

(*i*) Factor of a natural number ; (*ii*) Multiple of a natural number.

Factor. Multiple. If *a is divisible by b*, we say that *b is a factor of a* or that *a is a multiple of b.*

Thus,

b is a factor of a $\Leftrightarrow$ *a is a multiple of b.*

A factor is also often called a **Divisor** so that we have

a is a multiple of b $\Leftrightarrow$ *b is a divisor of a.*

Therefore, the symbol

$$b \mid a$$

denoting that a is divisible by b may as well be read as

(i) b is a factor of a

(ii) b is a divisor of a

(iii) a is a multiple of b.

To avoid confusion and help fixation of ideas, we shall read the expression

$$b \mid a.$$

as *b is a factor of a.*

Note. The student may recall that in the first chapter, he was introduced to the relations 'Is greater than' in the set of natural numbers. Here he is being introduced to another relation '*Is a factor of*' in N. Symbolically, the relation 'Is greater than' was denoted by '$>$' and we are now denoting the relation 'Is a factor of' by '|'.

In case b is not a factor of a, we shall write in symbols

$$b \nmid a.$$

The student is warned against any possible confusion of the symbol '$\nmid$' for 'is not a factor of' with the symbol '+' indicating the sum. In the symbol '$\nmid$' for 'is not a factor of' horizontal line does not cut the vertical line in the middle.

In the light of the discussion above, we have

$$3 \mid 6,\ 4 \mid 12,\ 5 \mid 15,\ 3 \mid 3,\ 1 \mid 3$$

and

$$4 \nmid 6,\ 5 \nmid 12,\ 6 \nmid 15,\ 3 \nmid 2,\ 3 \nmid 4.$$

Example

Find the set of factors of the numbers 18 and 7.

It may be easily seen that the set of factors of 18 is

$$\{1, 2, 3, 6, 9, 18\}$$

and the set of factors of 7 is

$$\{1, 7\}.$$

EXERCISES

1. Which of the following statements are true ?

(*i*) $15 \mid 45$ (*ii*) $36 \mid 12$ (*iii*) $15 \nmid 25$
(*iv*) $1 \mid 27$ (*v*) $23 \mid 1$ (*vi*) $11 \nmid 33$
(*vii*) $19 \nmid 38$ (*viii*) $23 \mid 69$ (*ix*) $14 \mid 56$
(*x*) $15 \nmid 27$.

2. Find the set of factors of each of the following natural numbers.

(*i*) 12 (*ii*) 48 (*iii*) 100
(*iv*) 41 (*v*) 125 (*vi*) 71
(*vii*) 300 (*viii*) 61 (*ix*) 123
(*x*) 240.

3. Put down any eight natural numbers and the set of factors of each of them.

4. Put down the set of multiples of each of the following natural numbers.

(*i*) 2 (*ii*) 5 (*iii*) 7
(*iv*) 6 (*v*) 3 (*vi*) 4
(*vii*) 11 (*viii*) 9 (*ix*) 10
(*x*) 1.

5. Put down the least member and the greatest member, in case it exists, of each of the sets of Ex. 2 and Ex. 4.

Observations. 1. We see that the set of factors of a number always has the least, namely the number 1, and the greatest the number itself. Such a set is always finite.

2. The set of multiples of a number has the least, namely the number itself, but it has no greatest. In contrast to the set of factors, which is finite, the set of multiples is in finite.

3. In view of the observation 1, we see that no factor of a number greater than the number itself. Below, we state and prove this result formally.

Theorem. No factor of a number is greater than the number itself.

In symbols,

$$a \mid b \Rightarrow a \leqslant b.$$

Proof. The symbol $a \leqslant b$ means that a is less than or equal to b *i.e.*, a is not greater than b. We have $a \mid b$ and as such there exists a number c such that

$$b = ac.$$

Let, if possible $\quad a > b.$

Now
$$a > b \Rightarrow ac > bc$$
$$\Rightarrow b > bc$$
$$\Rightarrow b \cdot 1 > bc$$
$$\Rightarrow 1 > c.$$

But $1 > c$ is impossible as 1 is the least natural number. So we arrive at a contradiction and, therefore, we have

$$a \leqslant b.$$

The relation 'Is factor of'. Given any two natural numbers a, b, we have

either $\quad a$ is a factor of b

or $\quad a$ is not a factor of b

i.e., in symbols we have

$$\text{either} \quad a \mid b \quad \text{or} \quad a \nmid b.$$

Thus, we have defined here a relation between pairs of natural numbers or as we may say a binary relation in the set of natural numbers. We say that '*Is a factor of*' is a relation in the set N of natural numbers. We shall study, in the following, the properties of the relation 'Is a factor of' as we had earlier studied those of the relation 'Is greater than'. But before doing so, we examine the following question :

Is there a pair of numbers such that each member of the pair is a factor of the other ?

Let us consider the pair of numbers 3 and 3. Now, we know that each of these is a factor of the other, because

$$3 \cdot 1 = 3.$$

In fact, if a is any natural number, each member of the pair (a, a) is a factor of the other, so if we put to ourselves the question 'Do there exist one or more of such pairs of different numbers ?', the answer is that there do not exist such pairs of natural numbers.

Properties of the Relation 'Is a factor of'

1. *Every natural number is a factor of itself and 1 is a factor of every natural number.*

If a is any natural number whatsoever, we have

$$a = a \cdot 1 \Rightarrow \begin{cases} a \mid a \\ 1 \mid a \end{cases}.$$

The property that *every natural number is a factor of itself* is expressed by saying that the relation '*Is a factor of*' *in the set* N *of natural numbers is reflexive.* This nomenclature '*Reflexive*' is justifiable because of each natural number bearing this relation to itself.

Cor. The only factor of the number 1 is 1 itself.

2. If a, b, c are three natural numbers such that a is a factor of b and b is a factor of c, then a is a factor of c.

$$a \mid b \text{ and } b \mid c \Rightarrow a \mid c.$$

Proof. As a is a factor of b, there exists a natural number d such that

$$b = ad. \qquad \ldots(1)$$

Again, as b is a factor of c, there exists a natural number e such that

$$c = be. \qquad \ldots(2)$$

Now, (1) and (2) imply that

$$c = (ad)\, e = a\, (de) \qquad \ldots(3)$$

and (3) implies that a is a factor of c.

Hence the result.

[The proof cauld as well be presented in the following form where we make rather heavier use of symbols.

$$\left.\begin{matrix} a \mid b \Rightarrow \exists\ d : b = ad \\ b \mid c \Rightarrow \exists\ e \cdot c = be \end{matrix}\right\} \Rightarrow c = (ad)\, e \Rightarrow c = a\,(de) \Rightarrow a \mid c.]$$

Transitivity of the Relation 'Is a factor of'. In view of the property referred to above, we say that the relation *'Is a factor of' in the set* N *of natural numbers is transitive.* This nomenclature is justifiable because the relation 'Is a factor of' is being transferred from one natural number to another.

Illustrations

(*i*) $3 \mid 6$, $6 \mid 12 \Rightarrow 3 \mid 12$

(*ii*) $5 \mid 15$, $15 \mid 60 \Rightarrow 5 \mid 60$

(*iii*) $8 \mid 32$, $32 \mid 96 \Rightarrow 8 \mid 96$.

The truth of each of these implications, which is of course, a consequence of the transitivity of the relation 'Is a factor of' may as well be directly verified.

3. *If a is a factor of b and b is a factor of a, then a and b are equal. In symbols*

$$a \mid b,\ b \mid a \Rightarrow a = b.$$

Proof. As a is a factor of b, there exists a natural number c such that

$$b = ac \qquad \ldots(1)$$

Again, as b is a factor of a, there exists a natural number d such thet

$$a = bd. \qquad \ldots(2)$$

Now (1) and (2) imply

$$\begin{aligned} b = (bd)\, c \Rightarrow b &= b\,(dc) \\ \Rightarrow b \,.\, 1 &= b\,(dc) \\ \Rightarrow 1 &= dc. \end{aligned}$$

Now, $1 = dc$ implies that c and d are factors of 1. But the only factor of 1 is the number 1 itself. Therefore, we have

$$c = 1, d = 1.$$

From (1) or (2), then we have

$$a = b.$$

The proof could as well be exhibited as follows :

$$\left.\begin{array}{l} a \mid b \Rightarrow \exists\, c : b = ac \\ b \mid a \Rightarrow \exists\, d : a = bd \end{array}\right\} \Rightarrow b = (bd)\, c$$

$$\begin{aligned} &\Rightarrow b \,.\, 1 = b\,(dc) \\ &\Rightarrow 1 \quad = dc \\ &\Rightarrow c \mid 1 \;\; , d \mid 1 \\ &\Rightarrow c = 1, d = 1 \end{aligned}$$

and so $$a = b.$$

Anti-symmetry of the Relation 'Is a factor of'. In view of the property of the relation 'Is a factor of' 'referred to above, we say that the relation 'Is a factor of' is anti-symmetric in the set N of natural numbers.

Example

Show that the relation '$\geqslant$' in N is anti-symmetric.

Here $a \geqslant b$ means either $a > b$ or $a = b$

Proof. $a \geqslant b \Rightarrow$ either $a > b$ or $a = b$

and $b \geqslant a \Rightarrow$ either $b > a$ or $b = a$.

Thus $a \geqslant b$ and $b \geqslant a \Rightarrow$ (either $a > b$ or $a = b$) and (either $b > a$ or $b = a$).

It is easy to see that none of the following is possible.

(*i*) $a > b$ and $b = a$

(*ii*) $a = b$ and $b > a$

(*iii*) $a > b$ and $b > a$.

In fact, it is an immediate consequence of the Trichotomy Law. Thus, when $a \geqslant b$ and $b \geqslant a$ we are left with the only possibility ($a = b$ and $b = a$).

$$\therefore \qquad a \geqslant b, b \geqslant a \Rightarrow a = b.$$

EXERCISE

Show that '$\leqslant$' is an anti-symmetric relation in the set of natural numbers. Here $a \leqslant b$ means either $a < b$ or $a = b$.

13. CRITERIA OF DIVISIBILITY BY 2, 3, 4, 5, 6, 8, 9, 10, 11.

In this section we try to discuss the criteria of divisibility of natural numbers by 2, 3, 4, 5, 6, 8, 9, 10, 11. Even though the treatment is purely through examples, yet an attempt has been made to explain to the reader the justification for the rules with which he might have had an acquaintance even earlier, and to help him to feel

at home with the correctness of the criteria. It may as well be remembered that the lines of the formal proof are exactly the same as in the following examples.

The basic result which enables us to have these rules is that 'If a natural number a is a factor of any two of the three natural numbers b, c and $b + c$, then it is a factor of the third. This result is, however, an immediate consequence of the notion of a factor and the distributive law.

In symbols we may write the result in the form :

(*i*) $a \mid b, a \mid c \Rightarrow a \mid (b + c)$

(*ii*) $a \mid b, a \mid (b + c) \Rightarrow a \mid c$

(*iii*) $a \mid c, a \mid (b + c) \Rightarrow a \mid b$.

In fact, we have

$$\left.\begin{array}{l} a \mid b \Rightarrow \exists\, d \,.\, b = ad \\ a \mid c \Rightarrow \exists\, e \cdot c = ae \end{array}\right\} \Rightarrow b + c = ad + ae$$

$$\Rightarrow b + c = a\,(d + e)$$

$$\Rightarrow a \mid (b + c).$$

Again, $$\left.\begin{array}{ll} a \mid b & \Rightarrow \exists\, d \,.\, b = ad \\ a \mid (b + c) & \Rightarrow \exists\, e : (b + c) = ae \end{array}\right\} \Rightarrow \{\, (b + c) - b \,\} = ae - ad$$

$$\Rightarrow c = a\,(e - d)$$

$$\Rightarrow a \mid c$$

The reader is advised to see the truth of the statement (*iii*) in a similar manner.

For example, 7 is a factor of 14 as also a factor of 21, and as such 7 is a factor of (14 + 21) *i.e.*, of 35. Again, 7 is a factor of 14 as also a factor of 49, *i.e.*, of (14 + 35) and as such 7 is a factor of 35.

I. Divisibility by 2. Let us consider the natural number

$$3528.$$

We can rewrite this number in the form

$$352 \times 10 + 8. \qquad \ldots (1)$$

Now, we, know that 2 is a factor of 10 and as such 2 will also be a factor of 352×10. Therefore 2 will be a factor of the number 3528 if and only if 2 is a factor of 8. But we know 2 is a factor of 8.

Thus, 2 is a factor of 3528.

In order to examine whether a given number is divisible by 2 or not, all that we have to see is whether the unit digit is divisible by 2 or not. Thus, a number is divisible by 2 if and only if its unit digit is either 2, 4, 6, 8 or zero.

We may note here that in case the last digit is zero, the number is divisible by 2, as it will be having 10 as its factor. As for example

$$3520 = 352 \times 10.$$

EXERCISE

Which of the following numbers are divisible by 2 ?

(*i*) 23	(*ii*) 306	(*iii*) 235
(*iv*) 356	(*v*) 5040	(*vi*) 7132
(*vii*) 3721	(*viii*) 3009	(*ix*) 1138
(*x*) 93244.		

II. Divisibility by 4. Consider the number

$$30778.$$

It can be rewritten as

$$307 \times 100 + 78. \qquad \ldots(2)$$

Now, we know that $4 \mid 100$ and so 4 is also a factor of 307×100. Therefore. in order to examine if the given number is divisible by 4 we have to see whether or not the number 78 is divisible by 4. In fact, we know that 78 is not divisible by 4, So the given number is not as well divisible by 4.

So, in order to examine whether a given number is divisible by 4 or not, it is sufficient to examine the divisibility of the number formed by the last two digits by 4, as for example in the illustration above.

EXERCISES

1. Rewriting the number in the form (2) above, examine which of the following numbers are divisible by 4.

(*i*) 5434	(*ii*) 4256	(*iii*) 2330
(*iv*) 9786	(*v*) 5004.	

2. Examine, which of the following statements are true.

(*i*) $4 \nmid 9786$	(*ii*) $4 \mid 7838$	(*iii*) $4 \mid 2780$
(*iv*) $4 \nmid 864324$	(*v*) $4 \mid 11128$	(*vi*) $4 \nmid 57896$

III. Divisibility by 8. Consider the number 213456. We can rewrite this number in the form

$$213 \times 1000 + 456. \qquad \ldots(3)$$

As 8 is a factor of 1000, we have that 213×1000 is divisible by 8. The given number, therefore, is divisible by 8 if and only if the number 456 is divisible by 8. Also we see that the number 456 is divisible by 8 and so the given number is divisible by 8,

Thus, any given natural number will be divisible by 8 if and only if the number formed by the last three digits, as 456 in the case of the example above, is divisible by 8.

EXERCISE

Which of the following statements are true ?

(*i*) $8 \mid 26$	(*ii*) $8 \mid 4328$	(*iii*) $8 \nmid 3248$
(*iv*) $8 \mid 3184$	(*v*) $8 \mid 453266$	(*vi*) $8 \nmid 432024$
(*vii*) $8 \mid 123312$	(*viii*) $8 \mid 255516$	(*ix*) $8 \mid 751364$
(*x*) $8 \nmid 2156304$.		

IV. Divisibility by 10. Any given natural number will have 10 as a factor if and only if its last digit is zero. As for example 2340 is divisible by 10 but 2304 is not.

EXERCISE

Which of the following numbers are divisible by 10 ?

(*i*) 3490	(*ii*) 1000	(*iii*) 2483
(*iv*) 2585	(*v*) 4230.	

V. Divisibility by 5. Consider any natural number, say 235773.

It may be written in the form

$$23577 \times 10 + 3. \qquad \ldots(5)$$

Now, 10 is divisible by 5 and so the number 23577×10 is divisible by 5.

So, the given natural number will be divisible by 5 provided 3 is divisible by 5. But $5 \nmid 3$ and so the given number is not divisible by 5.

In fact, we have in view of the above discussion that any given natural number is divisible by 5 if and only if its last digit is either 5 or zero.

EXERCISE

Examine whether the following statements are true or false.

(*i*) $5 \mid 325$	(*ii*) $5 \nmid 496$	(*iii*) $5 \nmid 700$
(*iv*) $5 \mid 234$	(*v*) $5 \mid 1250$	(*vi*) $5 \mid 3249$
(*vii*) $5 \nmid 5005$	(*viii*) $5 \nmid 1001$	(*ix*) $5 \mid 53005$
(*x*) $5 \nmid 509030$		

VI. Divisibility by 3. Let us consider the number,

354826.

The number can be rewritten as

$$3\,(99999 + 1) + 5\,(9999 + 1) + 4\,(999 + 1) + 8\,(99 + 1) + 2\,(9 + 1) + 6.$$

Using the Commutative and Associative laws of addition, we can put this number, finally in the form

$$[3 \times 99999 + 5 \times 9999 + 4 \times 999 + 8 \times 99 \times 2 \times 9] + [3 + 5 + 4 + 8 + 2 + 6]. \qquad \ldots(6)$$

Now, each of the numbers

9, 99, 999, 9999, 99999 ..

has 3 as a factor. Thus, the given number will be divisible by 3 if and only if the number

$$3 + 5 + 4 + 8 + 2 + 6$$

is divisible by 3.

Thus, we see that a given number is divisible by 3 if and only if the number obtained by summing up the digits is divisible by 3. In the present illustration the sum of the digits is 28 which is not a multiple of 3 and so the number is not divisible by 3.

Note. If some digits are zero, we omit the same while writing the sum of the digits.

EXERCISE

Express the following numbers in the form (6) above and state which of them are not divisible by 3.

(*i*) 2307	(*ii*) 4298	(*iii*) 23456
(*iv*) 9867	(*v*) 7083	(*vi*) 8735
(*vii*) 10578	(*viii*) 32178	(*ix*) 10305
(*x*) 32178		

VII. Divisibility by 9. As in the case of divisiblity by 3, a number, say,

$$34978$$

can be rewritten in the form

$$[3 \times 9999 + 4 \times 999 + 9 \times 99 + 7 \times 9] + 3 + 4 + 9 + 7 + 8. \qquad ...(7)$$

Now, each of the numbers written within brackets is divisible by 9 and so the given number is divisible by 9 if and only if,

$$3 + 4 + 9 + 7 + 8$$

is divisible by 9 *i.e.*, if and only if 31 is divisible by 9. But this is not so and, therefore, the given number is not divisible by 9.

Thus, a given number is divisible by 9 if and only if the number obtained by adding up the digits is divisible by 9.

The note to VI applies in this case as well.

EXERCISE

Write the following numbers in the form (7), above, and state which of these are divisible by 9.

(*i*) 34625	(*ii*) 38502	(*iii*) 325786
(*iv*) 149387	(*v*) 208575	(*vi*) 206037
(*vii*) 960209	(*viii*) 704256	(*ix*) 2505210
(*x*) 6403057.		

VIII. Divisibility by 6. Any number will be divisible by 6 if and only if it is divisible by 3 and 2 both. Thus, we have to apply both the tests of divisibility by 2 and 3 in order to examine divisibility by 6.

Thus, a given number, will be divisible by 6 if and only if, its last digit is one of 0, 2, 4, 6, 8 and the sum of the digits is divisible by 3.

EXERCISE

Which of the following statements are true and which are false ?

(*i*) $6 \mid 324$	(*ii*) $6 \mid 5301$	(*iii*) $6 \nmid 40744$
(*iv*) $6 \nmid 78000$	(*v*) $6 \nmid 73501$	(*vi*) $6 \nmid 60372$
(*vii*) $6 \nmid 92057$	(*viii*) $6 \mid 74582$	(*ix*) $6 \mid 827430$
(*x*) $6 \mid 85067352$		

IX. Divisibility by 11. Consider the number

$$745843$$

We may rewrite this number in the form

$$7\,(100001 - 1) + 4\,(9999 + 1) + 5\,(1001 - 1) + 8\,(99 + 1) + 4\,(11 - 1) + 3$$

or $$7\,(9091 \times 11 - 1) + 4\,(909 \times 11 + 1) + 5\,(91 \times 11 - 1) + 8\,(9 \times 11 + 1) + 4\,(11 - 1) + 3.$$

so that finally we have the number written as

$$[7 \times 9091 \times 11 + 4 \times 909 \times 11 + 5 \times 91 \times 11 + 8 \times 9 \times 11 + 4 \times 11] - [(7 + 5 + 4) - (4 + 8 + 3)]. \qquad \ldots(9)$$

Now each of the numbers in the first bracket is divisible by 11 so that the given number will be divisible by 11 if and only if the number $(7 + 5 + 4) - (4 + 8 + 3)$ is divisible by 11. This being not so, we conclude that the given number is not divisible by 11.

Thus, a given number is divisible by 11 if and only if the difference of the greater and the smaller of the numbers obtained on adding the alternate digits separately is divisible by 11. Also, the number will be divisible by 11 when the two sums of alternate digits are equal.

EXERCISES

1. Expressing the numbers in the form (9) above, examine which of them are divisible by 11.

(*i*) 704	(*ii*) 587	(*iii*) 2984
(*iv*) 8569	(*v*) 5985	(*vi*) 6017
(*vii*) 17592	(*viii*) 38986	(*ix*) 420409
(*x*) 735483.		

2. Which of the following statements are true and which are false ?

(*i*) $8 \mid 354078$	(*ii*) $9 \nmid 4593708$	(*iii*) $6 \nmid 4578004$
(*iv*) $11 \nmid 5485321$	(*v*) $4 \mid 5783486$	(*vi*) $9 \nmid 2874125$
(*vii*) $11 \mid 705349$	(*viii*) $6 \mid 503874$	(*ix*) $4 \nmid 9407382$
(*x*) $8 \nmid 1509344$		

14. PRIME NUMBERS. COMPOSITE NUMBERS.

Let a be any natural humber different from 1. We have seen that whatever the natural number $a \neq 1$ may be, it has at least two different factors *viz.*, 1 and a itself. Now, it so happens that there exist natural numbers which have just the two factors, namely 1 and the uumber itself. For example, consider the natural number

$$11.$$

It has no factor other than 1 and 11.

Of course, there are natural numbers a which have factors besides 1 and a. For example, let

$$a = 12.$$

Then besides 1 and 12, this natural number has other factors also. They are

$$2, 3, 4, 6.$$

These considerations lead us to the following definitions.

Prime Numbers

Definition. *A natural number different from 1 is said to to be prime if it has no factors other than 1 and itself.*

For example

$$2, 3, 5, 7, 11$$

are prime numbers.

Composite Numbers

Definition. *A natural number different from 1 is said to be composite if it is not prime.*

Thus a natural number is composite if it is different from 1 and has at least three different factors.

For example,

$$4, 6, 8, 9, 10, 12$$

are composite numbers.

It follows that given a natural number, a, we have one and only one of the following possibilities

(*i*) $a = 1$, (*ii*) *a is prime*, (*iii*) *a is composite.*

EXERCISES

1. Put down ten natural numbers which are prime.

2. Write twelve natural numbers which are composite.

3. Is there a natural number which is neither prime nor composite ? Is such a number unique ? Put down all those natural numbers which are neither prime nor composite.

4. Are all prime numbers odd ?

5. Write the set of all those primes which are even.

6. State whether the following statement is true or false
"There exists one and only one even prime number."

7. Give a list of all primes less than or equal to

(*i*) 100 (*ii*) 500 (*iii*) 1000.

8. Give the number of prime numbers less than or equal to the natural number n where n, has any of the following values

(*i*) 1 (*ii*) 2 (*iii*) 5
(*iv*) 10 (*v*) 14 (*vi*) 30.

9. The product of all natural numbers less than or equal to a given natural number n is called *Factorial n* and is denoted by

$$n\,!$$

For Exomple,

$$\begin{aligned} 1\,! &= 1 \\ 2\,! &= 1\,.\,2 &&= 2 \\ 3\,! &= 1\,.\,2\,.\,3 &&= 6 \\ 4\,! &= 1\,.\,2\,.\,3\,.\,4 &&= 24 \\ 5\,! &= 1\,.\,2\,.\,3\,.\,4\,.\,5 &&= 120. \end{aligned}$$

Verify that p is a factor of $(p-1)\,!+1$, for the following prime values of p.

(*i*) 2 (*ii*) 3 (*iii*) 5
(*iv*) 7 (*v*) 11 (*vi*) 13.

Note. In further studies, at the college level, the student will prove that p is a factor of $(p-1)\,!+1$ whatever prime number p may be. He will also prove that p is prime only if it is a factor of $(p-1)\,!+1$ Here he is only verifying the truth of the statement for some specific values of the prime number p

10. Verify that p is not a factor of $(p-1)\,!+1$ for the following composite values of p.

(*i*) 4 (*ii*) 6 (*iii*) 8
(*iv*) 9 (*v*) 10 (*vi*) 12.

The following two questions are of natural interest.

(*i*) *Is the set of prime numbers finite or infinite ?*

(*ii*) *Is the set of composite numbers finite or infinite ?*

The reply to the second question, as will be easily seen, is that

the set of composite numbers is infinite.

In fact, if we take any number, say 4, then each of the members of the infinite set

$$\{4^n : n \in \mathbf{N}\} \qquad ...(1)$$

is composite. This set consists of different powers of 4. Of course, lest there be any confusion in this regard, we state that (1) is not the set consisting of *all* composite

numbers. In fact, the set of composite numbers which do not belong to the set (1) is itself infinite.

It is interesting to learn that *the set of prime numbers is also infinite.* We shall give the proof of this important result a little later. As a consequence of the set of prime numbers being infinite, it will follow that there is a prime number greater than any given prime number. Thus, we say that the set of primes is

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23......\},$$

the dots indicating that there are prime numbers greater than 23 as well.

Theorem. Every natural number other than 1 admits of a prime factor.

Proof. Suppose that $a \neq 1$ is any natural number. We shall show that there exists a prime number which is a factor of a.

Now if a is itself a prime number, the theorem is proved inasmuch as the prime number a is a factor of itself.

Suppose now that a is a composite number. Being composite, it admits of a factor, say b, other than 1 and itself, *i.e.*,

$$b \mid a, \; b \neq 1, b \neq a.$$

If b is prime, we have finished. In the alternative case, there exists c, other than 1 and b such that

$$c \mid b.$$

Of course, we have

$$c < b, b < a.$$

If c is prime, we have again finished. In the alternative case, there will exist d, other than 1 and c such that

$$d \mid c.$$

Again,

$$d < c < b < a.$$

Now the possibility of this alternative cannot go on indefinitely and we will arrive, after a finite number of steps, at a number which is itself prime. For the fixation of ideas, suppose that, we have

$$f \mid e, e \mid d, d \mid c, c \mid b, b \mid a$$

and f is prime. The relation 'Is a factor of' being transitive, it follows that $f \mid a$ and f is prime.

Note. The point of the proof lies in our successively determining a factor of the factor obtained at the previous step and in noticing that after a finite number of steps we shall be arriving at a prime factor. For example, consider

$$a = 400.$$

Of the several factors of 400, we select any one, say 100 and write

$$b = 100.$$

Of the several factors of 100, we select any one, say 20 and write

$$c = 20.$$

Of the different factors of 20, we select, say, the factor 4 and write

$$d = 4.$$

Finally we see that 2 is a prime factor of $d = 4$. Thus, we have a chain of statements

$$2 \mid 4,\ 4 \mid 20,\ 20 \mid 100,\ 100 \mid 400$$

implying that

$$2 \mid 400$$

because of the transitivity of the relation 'Is a factor of'

Of course at each step, we could exercise any choice of a factor so that different choices at different stages may lead us to different prime factors of the given number. Thus in relation to the given number 400, we could as well have the following chain of divisibility relations

$$5 \mid 25,\ 25 \mid 100,\ 100 \mid 400$$

showing that the prime number 5 is a factor of 400.

The reader may try his hand at this technique with different numbers, say,

162, 375, 399.

Theorem. The set of prime numbers is infinite.

Proof. We suppose that the statement is false, *i.e.*, we suppose the negation of this statement *viz.*,

'The set of prime numbers is not infinite'

or equivalently

'The set of prime numbers is finite,'

is true.

The set of prime numbers being finite, there exists the greatest prime number. We suppose that q is the greatest prime number.

Consider the product of all the prime numbers, *viz.*, the number

$$b = 2 \cdot 3 \cdot 5 \quad 7 \ \cdot \ \ldots\ldots q. \qquad \ldots(1)$$

We write

$$a = b + 1 \qquad \ldots(2)$$

so that the number a is one more than the product of all the prime numbers. Surely

$$a \neq 1.$$

The number a must have a prime factor. Suppose p is a prime factor of a. Surely p is one of the numbers

$$2, 3, 5, 7 \ldots\ldots\ldots q$$

involved in the product (1). We have

$$p \mid a \text{ and } p \mid b$$

implying that

$$p \mid (a - b).$$

As $a - b = 1$, we conclude that

$$p \mid 1$$

i.e., p is a factor of 1.

No prime number, however, is a factor of 1, because the only factor of 1 is the number 1. Thus, we arrive at a false statement. Therefore, the statement 'the number of prime numbers is not infinite' must not be true. It follows that the set of the prime numbers is infinite.

Note. Having proved that the set of prime numbers is infinite, it follows that there exist prime numbers greater than any given prime number. Thus the list

2, 3, 5, 7, 11, 13......

of prime numbers is endless. What we shall naturally do is to go on selecting out of the list

1, 2, 3, 4, 5, 6......

of natural numbers those which are prime. It should be seen, however, that the process of determining whether or not a given natural number is prime becomes more and more laborious as the numbers increase.

Thus, for example, it may be quite an uphill task to decide whether or not the number

3570397643

is prime. In fact quite challenging problems have ofen been proposed by mathematicians asking for the proof of the primeness of the number proposed.

15. HIGHEST COMMON FACTOR.

We introduce the notion of the highest common factor of two natural numbers by means of an example.

Consider the two natural numbers

45, 63.

The sets of factors of these two numbers are

{1, 3, 5, 9, 15, 45}
{1, 3, 7, 9, 21, 63}

respectively The intersection of these two sets is the set

{1, 3, 9}

of common factors of the given numbers.

Finally the higest of the members of this set of common factors is 9. This number 9 is referred to as the *Higest Common Factor* abbreviated as HCF of the two numbers 45, 63.

Let us consider a second illustration.
Let 12, 20 be two given natural numbers.

The sets of factors of these two numbers are

{1, 2, 3, 4, 6, 12}, {1, 2, 4, 5, 10, 20}

and the set of common factors of the two numbers being the intersection of the sets of the factors, is

{1, 2, 4}.

Now, 4 being the highest of the members of the set of common factors, we see that 4 is the HCF of the numbers 12, 20.

EXERCISE

Carry out the procedure outlined above for the following pairs of natural bumbers and find their HCF

(*i*) 36, 64	(*ii*) 30, 135	(*iii*) 28, 56
(*iv*) 21, 98	(*v*) 16, 45	(*vi*) 84, 128.

Highest Common Factor of two Numbers

Definition. *The greatest of the common factors of two numbers is called their Highest Common Factor.*

The highest common factor of two numbers is often abbreviated as HCF.

The procedure outlined above in respect of specific pairs of natural numbers indicates that *any two given natural numbers have an HCF, and that the same is unique.*

Let a, b be two given natural numbers. Let A,B denote the sets of factors of the numbers a, b respectively. Then

$$A \cap B$$

denotes the set of the common factors of the numbers a, b. Here, A and B denote what we had in the previous chapter written as Fac a, Fac b.

Now the sets A, B of numbers a, b are both finite. It follows that their intersection

$$A \cap B$$

is also finite. Moreover, this intersetion is not a void set. In fact we know of atleast one natural number 1, which, belonging as it does to A as well as B, must itself belong to

$$A \cap B.$$

Thus we see that the set $A \cap B$ of common factors is a non-empty finite set. It must, therefore, have a greatest member and this, greatest number is, by definition, the unique HCF of a and b.

We have thus shown that any two given natural numbers admit of a unique highest common factar.

Note. The theorem which we have proved above is of thereoretical importance inasmuch as given any two natural numbers, we are, as a result of the therorem, confident that they do have HCF and that, by whatever possible process one would compute it, the results are going to be indentical. The question now relates to our computing the HCF of any two given natural numbers. Surely when the numbers are not very large, we can carry out the process followed in the proof which is the one followed earlier for finding HCF of the pairs

(*i*) 45, 63 and (*ii*) 12, 20.

Clearly this procedure would become very elaborate when the natural numbers, in whose HCF we are interested, are very large.

Luckily, however, there is a simpler procedure available for finding the HCF of any two given numbers. This process of repeated division for finding the HCF of two numbers, called 'Euclid's Algorithm' appeared in Euclid's Elements about 2300 years ago. We now proceed to describe this process.

Algorithm for the Determination of HCF.

Let $$a, b$$
be two given natural numbers and let $a > b$.

Now if b is itself a factor of a, then the HCF of a, b is b because the highest of the factors of b, which is b is also a factor of a.

Suppose that b is not a factor of a.

By the division algorithm, there exist natural numbers q, r such that

$$a = bq + r, r < b. \quad \ldots(i)$$

We now show that

the HCF of a and b

is the same as

the HCF of b and r.

This will be so if the set of common factors of a and b is as well the set of common factors of b and r, *i.e.*, a common factor of a and b is a common factor of b and r and *vice versa.*

Let x be a common factor of a and b. There exist, therefore, two natural numbers u, v such that

$$a = xu, b = xv. \quad \ldots(ii)$$

From (i), and (ii), we have

$$xu = xvq + r$$
$$\Rightarrow \quad r = x(u - vq)$$
$$\Rightarrow \quad x \text{ is a factor of } r.$$

It follows that x, a common factor of a and b, is as well a common factor of b, r.

Now, suppose that y is a common factor of b and r. There exist, therefore, two natural numbers s, t such that

$$b = ys, r = yt. \quad \ldots(iii)$$

From (i) and (iii), we have

$$a = ysq + yt = y(sq + t)$$
$$\Rightarrow \quad y \text{ is a factor of } a.$$

It follows that y, a common factor of b and r is as well a common factor of a, b.

Thus we see *that the HCF of a and b is also the HCF of b and r, where r is the remainder obtained on dividing a by b.*

This important principle gives us the necessary clue for computing the HCF of two given numbers.

We apply to the pair (b, r) the process we applied to the pair (a, b) in that we divide b by r. If r_1 denotes the remainder obtained on dividing b by r, we see as above that the HCF of r, r_1 is the same as that of b, r and as much as that of a, b.

We notice that $r_1 < r$.

In case r_1 is a factor of r, the HCF of r_1, r being r_1, it follows that the HCF of a, b is r_1. If, however, r_1 is not a factor of r, we again divide r by r_1 and obtain the remainder, say r_2 so that $r_2 < r_1$. As the remainders go on decreasing, the process must end after a finite number of steps *i.e.*, we shall obtain a remainder h which is a factor of the previous remainder, say k. The HCF of h and k will be h and as it is the same as the HCF of a and b, we have that the HCF of a and b is h.

This process is illustrated below : Consider the two numbers

15844, 13281.

We have as a result of successive divisions

$$15844 = 13281 \times 1 + 2563$$
$$13281 = 2563 \times 5 + 466$$
$$2563 = 466 \times 5 + 233$$
$$466 = 233 \times 2$$

and as such 233 is the HCF of the given numbers, being the last remainder which is a factor of the remainder 466 previous to it.

The process may be arranged as follows :

	2	5	5	1
	466	2563	13281	15844
233	466	2330	12815	13281
		233	466	2563

2. Consider the numbers

1404, 1014.

We have

$$1404 = 1014 \times 1 + 390$$
$$1014 = 390 \times 2 + 234$$
$$390 = 234 \times 1 + 156$$
$$234 = 156 \times 1 + 78$$
$$156 = 78 \times 2.$$

The last remainder 78 which is a factor of the previous remainder 156 is the required HCF.

The process may be exhibited as follows :

	2	1	1	2	1
	156	234	390	1014	1404
78	156	156	234	780	1014
		78	156	234	390

EXERCISE

Find the HCF of the following pairs of numbers :

(*i*) 15087, 10857 (*ii*) 9154, 3781

(*iii*) 1375, 4935 (*iv*) 3696, 6300.

Let a, b be two given natural numbers and let h denote their HCF. A, B, H denote the sets of factors of a, b, h respectively.

It is clear that every factor of h is as well a factor of a, b, *i.e.*, every factor of the HCF, h of a, b is a common factor of a, b.

In fact let d be a factor of h so that we have

$$d \mid h.$$

We also have

$$h \mid a \quad \text{and} \quad h \mid b.$$

Now

$$d \mid h, h \mid a \Rightarrow d \mid a$$
$$d \mid h, h \mid b \Rightarrow d \mid b$$

so that it follows that every factor of the HCF h of a, b is a common factor of a, b. In terms of set notation, we have

$$H \subset (A \cap B). \qquad ...(1)$$

We shall now show that the statement

$$(A \cap B) \subset H \qquad ...(2)$$

is as well true *i.e.*, every common factor of a, b is as well a factor of their HCF, h.. As a result of the two statements (1) and (2) it will follow that

$$A \cap B = H$$

i.e., the set of common factors of a, b coincides with the set of factors of their HCF, h.

We thus state and prove the theorem as follows.

Theorem. Every common factor of two numbers is factor of their HCF.

Proof. We suppose that we perform successive divisions involved in the process of finding the HCF of a, b. Surely h will be the last remainder which will be

a factor of the previous remainder which we may denote by k. It follows that the set of common factors of a, b is the same as the set of common factors of k, h. Since h is a factor of k, we see that the set of factors of h is exactly the set of common factors of k, h and, therefore, the set of common factors of a, b.

It follows that

$$H = A \cap B$$

Illustrations

1. Let

$$a = 45, b = 63.$$

We have

$$\begin{aligned} A &= \{1, 3, 5, 9, 15, 45\} \\ B &= \{1, 3, 7, 9, 21, 63\} \\ A \cap B &= \{1, 3, 9\} \\ h &= 9 \\ H &= \{1, 3, 9\}. \end{aligned}$$

Clearly, we have

$$H = A \cap B.$$

2. Consider the numbers

$$a = 36, b = 64.$$

We have

$$\begin{aligned} A &= \{1, 2, 3, 4, 6, 9, 12, 18, 36\} \\ B &= \{1, 2, 4, 8, 16, 32, 64\} \\ A \cap B &= \{1, 2, 4\} \\ h &= 4 \\ H &= \{1, 2, 4\}. \end{aligned}$$

It is thus verified that

$$H = A \cap B.$$

EXERCISE

Carry out the procedure outlined above for the following pairs of natural numbers.

(*i*) 24, 72 (*ii*) 42, 55 (*iii*) 18, 99 (*iv*) 75, 40.

A Property of the HCF of two Numbers.

The HCF of

$$ma, mb$$

is the product by m of the HCF of

$$a, b$$

m being any given natural number.

Let h denote the HCF of a, b so that we have to show that mh is the HCF of ma, mb.

Before giving the proof, we consider a specific case of two numbers to pinpoint the idea involved in the proof.

Let

$$a = 45, b = 63$$

and

$$m = 4$$

so that the numbers ma, mb are

$$180, 252.$$

We obtain sets of successive remainders in the process of finding the HCF of the pair 45, 63 and the pair 180, 252.

In the following we exhibit the information in this regard :

I

	2	2	1
	18	45	63
9	18	36	45
		9	18

II

	2	2	1
	72	180	252
36	72	144	180
		36	72

We see that the number of entries in each of the two rows of remainders is the same and each entry in II is 4 times the corresponding entry in I. As such it follows that the HCF of 4×45, 4×63 is 4×9 ; 9 being the HCF of 45, 63.

The essential point in the proof is that the number of remainders in relation to ma, mb is the same as the number of remainders in relation to a, b and each remainder in the first case is m times the corresponding remainder in the second case.

We now give the proof.

Proof. Let

$$a = bq + r, r < b.$$

It implies

$$ma = m(bq + r)$$
$$= (mb)\, q + mr.$$

Also

$$r < b \Rightarrow mr < mb.$$

Thus, it follows that mr is the remainder obtained on dividing ma by mb.

It will follow that the remainder obtained on dividing mb by mr will be m times the remainder obtained on dividing b by r.

Thus, we see that the last remainder obtained in relation to ma and mb will be m times the last remainder in relation to a, b.

Hence the result.

Cor. *Let d be a common factor of a, b. Then the* HCF *of*

$$a \div d, b - d$$

is

$$h \div d$$

h being the HCF of a, b.

Surely $a \div d, b \div d$ are both natural numbers. Let h' be the HCF of $a \div d, b \div d$. Then, by the preceding theorem, $h'd$ is the HCF of $d(a \div d)$, $d(b \div d)$, *i.e.*, of a, b. Thus, we have

$$h'd = h \Rightarrow h' = h \div d.$$

Hence the corollary.

In particular it follows that 1 is the HCF of

$$a \div h, b \div h.$$

Illustrations

1. The HCF of

$$36, 60$$

is 12 and that of

$$3 \times 36, 3 \times 60$$

is

$$3 \times 12 = 36.$$

2. The HCF of

$$36, 60$$

is 12 and 2 being a common factor of 36 and 60 the HCF of

$$36 \div 2, 60 \div 2$$

is

$$12 \div 2 = 6.$$

3. The HCF of

$$36, 60$$

is 12 and that of

$$36 \div 12, 60 \div 12$$

i.e., of

$$3, 5$$

is

$$12 \div 12 = 1.$$

This means that 1 is the only common factor of 3, 5.

Highest Common Factor of More Than Two Numbers.

Having considered the case of two numbers, we proceed to show how the notion of HCF can be extended to that of any finite set of natural numbers. Since the idea involved in relation to any finite set of numbers is basically the same as in relation to three, we consider only the latter.

Let a, b, c be any three natural numbers and let A, B, C, be the sets of their factors.

Consider the intersection

$$A \cap B \cap C. \qquad \ldots(1)$$

Surely this intersection set is non-empty and finite, consisting as it does of the common factors of a, b, c.

The greatest of the members of the finite non-empty set (1) is the highest common factor of a, b, c. Thus, we say that *the highest common factor of three numbers is the greatest of the common factors of the three numbers.*

Surely this exists and is unique.

We give below a result which will indicate a practical method for determining the HCF of three or more numbers.

Theorem. The highest common factor of three numbers is the higest common factor of any one of them and the highest common factor of the other two.

Proof. Let a, b, c be three numbers

Let h denote the highest common factor of two of them, say, a, b. We have

$$A \cap B \cap C = (A \cap B) \cap C.$$

Also we know that

$$H = A \cap B$$

so that we have

$$A \cap B \cap C = H \cap C.$$

Now the greatest of the members of the set

$$A \cap B \cap C$$

is the HCF of a, b, c and the greatest of the members of the set

$$H \cap C$$

is the HCF of h and c.

It follows that the HCF of a, b, c is the HCF of c and the HCF, h of a, b.

EXERCISE

Find the HCF of

(*i*) 15807, 10857, 19024.

(*ii*) 3696, 6300, 9282.

16. CO-PRIMES. GAUSS'S THEOREM

Co-Priem. We have seen that the set of common factors of two numbers is a non-empty finite set inasmuch as 1 is always a member of this set of common factors Of course, this set of common factors would ordinarily have members other than 1. Sometimes, however, it so happens that the set of common factors of two given numbers has only one member, 1 *i.e.*, the only common factor of the two numbers is 1.

Let us consider some illustrations.

1. $$a = 12, b = 15$$
$$A = \{1, 2, 3, 4, 6, 12\} \qquad B = \{1, 3, 15, 5\}$$
$$A \cap B = \{1, 3\}.$$

2. $$a = 20, b = 9$$
$$A = \{1, 2, 4, 5, 10, 20\}, \qquad B = \{1, 3, 9\}$$
$$A \cap B = \{1\}.$$

We are led to the following definition.

Definition. *A pair of numbers is said to be co-prime, if the numbers have no common factor other than* 1.

The two numbers which are co-prime are also said to be relatively prime.

It is easy to see that two numbers are co-prime if and only if their highest common factor is 1.

Illustrations

The pairs of numbers (*i*) 12, 35 (*ii*) 63, 26 (*iii*) 162, 35

are co-prime and the pairs of numbers

(*i*) 6, 8 (*ii*) 45, 65 (*iii*) 36, 216

are not co-prime.

Caution. The reader is cautioned against any possible confusion between the concept of a prime number and that of a pair of numbers being co-prime Whereas, the first relates to a natural number, the second concerns a pair of natural numbers. The reader may as well see the truth of the following statements.

(1) Two prime numbers are always co-prime.

For example, 7 and 19 are co-prime.

(2) None, one both of a pair of co-prime numbers may be prime.

For example

(*i*) 12, 25 (*ii*) 12, 5 (*iii*) 11, 13

are pairs of co-primas in respect of which, none, one and both the numbers are primes respectively.

Theorem. The number h is the HCF of two numbers a and b if and only if h is a common factor of a and b and the two numbers $a - h$ and $b \div h$ are relatively prime.

Proof. Let h be the HCF of a and h. Then the HCF of

$$a \div h, b \div h \text{ is } h \div h = 1,$$

and, therefore,

$$a \div h \text{ and } b \div h \text{ are co-primes.}$$

Conversely, if h is common factor of a and b such that the HCF of $a \div h$ and $b \div h$ is 1, it follows that the HCF of

$$h\,(a \div h) \text{ and } h\,(b \div h) \text{ is } h.\,1$$

i.e., the HCF of a and b is h.

We have the result as stated.

EXERCISES

Which of the followiug pairs of numbers are co-prime ?

(*i*) 45, 63 (*ii*) 119, 299 (*iii*) 140, 91

(*iv*) 609, 2157 (*v*) 859, 1311 (*vi*) 315, 207.

2. There is a theorem known as Fermat's theorm which is stated as follows :

If p is any prime number and a is prime to p then p is a factor of $a^{p-1} - 1$, *i.e.*,

$$p \mid (a^{p-1} - 1).$$

Verify this theorem for the follwing pairs of values of a and p

(*i*) $a = 2, p = 3$ (*ii*) $a = 3, p = 5$ (*iii*) $a = 4, p = 3$

(*iv*) $a = 5, p = 3$ (*v*) $a = 6, p = 5$ (*vi*) $a = 5, p = 7$.

Gauss's Theorem

Before stating this theorem, we make a few observations.

Let c be a factor of a so that we have

$$c \mid a.$$

If now b be any number whatsoever, then

$$c \mid a \Rightarrow c \mid ab$$

i.e., if c is a factor of a, then it is as well a factor of ab.

In general, we see that if c is a factor of any of a or b, then c is as well a factor of ab *i.e.*,

$$c \mid a \text{ or } c \mid b \Rightarrow c \mid ab.$$

We are now naturally interested in the converse. We suppose that a, b, c are three numbers such that c is a factor of ab *i.e.*,

$$c \mid ab.$$

The question now relates to c being a possible factor of a or of b.

Let us have a few specific cases.

(1) Let

$$a = 6, b = 15, c = 10$$

so that while we have

$$c \mid ab \Leftrightarrow 10 \mid 90$$

neither c is a factor of a nor of b *i.e.*, in this case while we have

$$c \mid ab$$

we have

$$\text{neither} \quad c \mid a \text{ nor } c \mid b.$$

(2) Let

$$a = 12, b = 30, c = 6$$

so that we have

$$c \mid ab \Leftrightarrow 6 \mid 360$$

as also $$c \mid a \Leftrightarrow 6 \mid 12$$

and $$c \mid b \Leftrightarrow 6 \mid 30.$$

(3) Let

$$a = 12, \ b = 30, \ c = 10.$$

In this case we have $c \mid ab \Leftrightarrow 10 \mid 360.$

Also while we have

$$c \mid b$$

we do not have

$$c \mid a.$$

We have seen that we have had illustrations of each of the three conceivable possibilities. The following theorem gives us a very useful information pertaining to the problem in question.

Gauss's Theorem. *Let a, b, c be three numbers such that*

(*i*) *c is a factor of the product ab.*

(*ii*) *c is co-prime to a.*

Then c is a factor of b.

Proof. Now c, a being co-prime, their HCF is 1. It follows that the HCF of

$$cb, ab$$

is $$b \times 1 = b.$$

Now c is a common factor of the two numbers

$$cb, ab$$

whose HCF is b.

It follows that c is a factor of b [Refer Theorem on Pages 80-81].

Cor. If a prime p divides the product p_1p_2 of two primes, then it must be equal to at least one of p_1 and p_2. More generally, if a prime p divides the product of any number of primes, then it must be equal to at least one of them.

Illustration.

Let $$a = 12, b = 30, c = 5.$$

We have

$$c \mid ab \Leftrightarrow 5 \mid 360.$$

Also 5 is co-prime to 12 It follows by Guass's Theorem, as may also be directly seen, that 5 is a factor of 30.

17. LOWEST COMMON MULTIPLE

Let us consider two numbers 15 and 6. The sets of multiples of 15 and 6 are

$$\{1 \times 15, 2 \times 15, 3 \times 15, 4 \times 15 \ldots\}$$

and

$$\{1 \times 6, 2 \times 6, 3 \times 6, 4 \times 6, 5 \times 6 \ldots\}.$$

Both these sets are infinite. The intersection of these two sets consists of numbers which are multiples of 15 as well as 6. We may say that, this set is the set of common multiples of 15 and 6. This intersection set is not empty Its members are 30, 60, 90 ..., and so it will have a smallest member namely 30. This smallest member 30 is called the *Lowest Common Multiple* abbreviated as LCM of 15 and 6.

Instead of dealing with two particular numbers 15 and 6, we now deal with any two numbers a and b. Ths sets of multiples of a and b will be respectively

$$\{a, 2a, 3a \ldots\ldots\}$$

and

$$\{b, 2b, 3b \ldots\ldots\}.$$

We may denote these sets by $\{xa : x \in N\}$ and $\{xb : x \in N\}$ Consider the intersection set consisting of multiples of a and as well of b, It is non-empty because it has at least one element ab which is a multiple of both a and b.

This intersection set will have a smallest member. This smallest member of the set of common multiples of a and b is called the *lowest common multiple* of a and b, abbreviated as LCM of a, b.

Definition. *The smallest of the common multiples of two numbers is called their Lowest Common Multiple.*

Surely, the LCM of any two given numbers exists and is unique.

EXERCISES

1. Put down the sets of multiples of the following pairs of numbers and find their LCM.

(*i*) 8, 12 (*ii*) 9, 6 (*iii*) 4, 8
(*iv*) 14, 22 (*v*) 7, 11 (*vi*) 4, 14
(*vii*) 15, 20 (*viii*) 24 30 (*ix*) 8, 10
(*x*) 21, 24.

Note. 1 The intersection set of the sets of multiples of any three numbers a, b, c will be non-empty inasmuch as it will have at least the member abc. This set, therefore, will have a smallest member which is called the Lowest Common Multiple of a, b, c. Similarly, we may define the LCM of any finite set of numbers, as the smallest member of the intersection set of the sets of multiples of these numbers.

2 Using only the definition, find the LCM of the following sets of numbers.

(*i*) 2, 4, 10 (*ii*) 7, 6, 14 (*iii*) 5, 10, 15.

Note. 2 While ab is a common multiple of a and b, every multiple of ab is as well a multiple of a and b, so that

$$ab, 2ab, 3ab, \ldots$$

are also all multiples of a and b. These multiples of ab may not exhaust the set of common multiples of a and b. The reader may see this point clearly made out in the following illustration in which the set of common multiples of 15 and 6 is

$$(30, 60, 90, \ldots) \qquad \ldots (i)$$

but the set of multiples of 15×6 is

$$(90, 180, 270, \ldots). \qquad \ldots (ii)$$

Obviously the set (ii) is a sub-set of the set (i). In the following section, we prove that there exists a number, the set of whose multiples is the set of comon multiples of the two numbers. For the two numbers 15 and 6, this number comes out to be

$$\frac{15 \cdot 6}{3},$$

where 3 is the HCF of the two numbers.

**Theorem.* If h denotes the HCF of two given numbers a and b, then the set $\{x\,(ab \div h) : x \in \mathbf{N}\}$ of the multiples of $ab \div h$ is exactly the set of common multiples of a and b. In symbols, we have

$$\{x\,(ab \div h) : x \in \mathbf{N}\} = \{xa : x \in \mathbf{N}\} \cap \{xb : x \in \mathbf{N}\}.$$

Proof. h is the HCF of a and b.

There exist numbers a' and b' such that

$$a = ha' \text{ and } b = hb'.$$

Surely HCF of $a \div h$ and $b \div h$ will be $h \div h$, *i.e.*, HCF of a' and b' is 1.

Let u be some common muitiple of a and b. Then there exist two numbers c, d such that

$$u = ca \text{ and } u = db.$$

Also we have

$$a = ha' \text{ and } b = hb'.$$

Thus we obtain

$$u = cha' = h\,(ca') \text{ and } u = dhb' = h\,(db')$$

We obtain

$$h\,(ca') = h\,(db') \Rightarrow ca' = db'.$$

Again

$$ca' = db' \Rightarrow b' \mid (ca').$$

---

*The proof of this theorem and that of the Unique Factorisation Theorem may be omitted on first reading. However, the reader must acquaint himself with the contents of both these important theorems

Now b' is a factor of ca' and b', a' are relatively prime. By Gauss's theorem, it follows that

$$b' \mid c.$$

There exists, therefore, a natural number m such that

$$\begin{aligned} c &= b'm \\ \Rightarrow ca' &= (b'm)\, a' = m\,(b'a') \\ \Rightarrow u &= cha' \quad = hm\,(b'a') \\ &\qquad\qquad = mh\, b'a' \\ &\qquad\qquad = m\,(ab \div h). \end{aligned}$$

It follows that the common multiple u of a, b is a multiple of

$$(ab) \div h$$

and as such every common multiple of a and b is multiple of $(ab) \div h$.

Now we show that every multiple of $ab \div h$ is a multiple of a as well as b.

Consider any multiple

$$x\,(ab \div h)$$

of $\quad ab \div h.$

We have

$$\begin{aligned} x\,(ab \div h) &= xab \div h \\ &= xa\,(b \div h) \\ &= x\,(b \div h)\, a. \end{aligned}$$

Also $\quad x\,(ab \div h) = x\,(a \div h)\, b.$

It follows that every multiple of

$$ab \div h$$

is a multiple of a and of b.

Hence the theorem.

Theorem. The product of two numbers is equal to the product of their highest common factor and their lowest common multiple.

Proof. Let a, b be two given numbers and let h, l denote their highest common factor and lowest common multiple respectively.

We have to show that

$$hl = ab.$$

We have seen that the set of common multiples of a, b is the set

$$\{x\,(ab \div h) : x \in \mathrm{N}\}$$

so that the lowest common multiple of a, b is $ab \div h$ and we have

$$\begin{aligned} ab \div h &= l \\ \Rightarrow \qquad ab &= hl. \end{aligned}$$

Hence the result.

Note. This theorem gives us a method for finding the lowest common multiple of two given numbers.

Let a, b be two given numbers. Then their lowest common multiple is

$$(ab) \div h;$$

h being the highest common factor of a, b.

It also follows that every multiple of the lowest common multiple of two numbers is a multiple of each of the two numbers.

EXERCISE

Find the LCM of the following pairs of natural numbers.

(*i*) 420, 135 (*ii*) 252, 360 (*iii*) 16, 20.

18. UNIQUE PRIME FACTORISATION

We have already seen that every number other than 1 has a prime factor. We shall now sharpen this result and prove that every number is expressible as a product of primes. Thus for example, we have

$$210 = 2 \times 3 \times 7 \times 5$$

where each of the factors on the right hand side is a prime number. As another example we have

$$308 = 2 \times 2 \times 7 \times 11.$$

Not only is every number expressible as a product of primes, it is as well true that irrespective of how we express a given number as a product of primes, the prime factors will be exactly the same apart from the order of factors. Thus for example, we could as well write

$$210 = 7 \times 3 \times 2 \times 5,\ 210 = 3 \times 5 \times 2 \times 7, \text{etc.},$$

but the fact, as we shall prove, is that howsoever we may express 210 as a product of primes, we shall always obtain the same primes *viz.*, 2, 3, 5, 7.

Of course if any prime in any one decomposition occurs more than once, it will occur the same number of times in every decomposition. Thus, in every decomposition of 308 as a product of primes, the prime factor 2 will occur twice.

The reader is advised to verify the truth of the statement in respect of few numbers, say,

(*i*) 3146 (*ii*) 204 (*iii*) 1085 (*iv*) 101 (*v*) 442.

We now state and prove what is called the *unique prime factorisation theorem.*

Theorem. Every natural number, different from 1, is expressible as a product of primes and the expression is, apart from the order of factors, unique.

Proof. Let x be any given number. If x is prime, we have nothing else to prove. Now suppose that x is not prime. Therefore, it admits of a prime factor, say p_1, so that we have a result of the form

$$x = p_1 x_1, \quad x_1 < x.$$

If x_1 is prime, then we have proved the theorem. If, however, x_1 is not prime, we have, then, a result of the form

$$x_1 = p_2 x_2$$

where p_2 is prime and $x_2 > x_1$.

Proceeding in this manner, we obtain a sequence of primes

$$p_1, p_2, \ldots\ldots \qquad \ldots (1)$$

and a sequence of numbers

$$x_1\ x_2, \ldots\ldots \qquad \ldots (2)$$

such that

$$x > x_1 > x_2 \ldots\ldots$$

Because the sequence (2) is successively decreasing, this process must stop after a finite number of steps. So we must arrive at a member of the sequence (2) which is prime. Let us suppose that x_{n-1} is a prime number. We have, then

$$x = p_1 p_2 \ldots . p_{n-1}\ x_{n-1}.$$

Writing p_n for x_{n-1}, we have

$$x = p_1 p_2 \ldots\ldots p_{n-1} p_n \qquad \ldots (3)$$

(3) expresses x as a product of primes. These primes, of course, may not be all different.

Uniqueness of (3). Let, if possible,

$$x = q_1 q_2 \ldots . q_m \qquad \ldots (4)$$

be an alternative expression of x as a product of primes.

We suppose that $n \leqslant m$. We have from (3) and (4)

$$p_1 p_2 . \ldots . p_n = q_1 q_2 \ldots\ldots q_m \qquad \ldots (5)$$

Now (5) shows that the prime p_1 is a factor of the product,

$$q_1 q_2 \ldots . q_m,$$

and so p_1 must be equal to some one of these primes. Without any loss of generality, we suppose that $p_1 = q_1$. Such a supposition is possible because it only amounts to chainging the order of the factors and suitably renaming them.

Because $p_1 = q_1$ with the help of the cancellation law of multiplication, we have, from (5)

$$p_2 p_3 \ldots p_n = q_2 q_3 \ldots\ldots q_m . \qquad \ldots (6)$$

As before we have that p_2 must be equal to some one of the primes $q_2, q_3 \ldots\ldots q_m$ Without loss of generality, we suppose that $p_2 = q_2$ and so we have from (6)

$$p_2, p_4 \ldots p_n = q_3 q_4 \ldots\ldots q_m. \qquad \ldots (7)$$

Proceeding similarly, we have,

$$p_3 = q_3, p_4 = q_4, \ldots . \ldots p_n = q_n. \qquad \ldots (8)$$

and $$q_{n+1} q_{n+2} \ldots\ldots q_m = 1 \qquad \ldots (9)$$

if $$m > n.$$

But no prime is a factor of 1.

We have that $m > n$ leads to a contradiction.

$$\therefore \qquad m = n,$$

and so the two decompositions of x

$$p_1 p_2 p_3 \ldots . p_n$$

and

$$q_1 q_2 q_3 \ldots \ldots q_n$$

are identical.

EXERCISE

Express the following as products of prime factors.

(*i*) 675	(*ii*) 528	(*iii*) 990
(*iv*) 1024	(*v*) 660	(*vi*) 26000
(*vii*) 4050	(*viii*) 11220	(*ix*) 99792
(*x*) 874944.		

19. DETERMINATION OF HCF AND LCM OF TWO NUMBERS THROUGH EXPRESSION OF THE GIVEN NUMBERS AS PRODUCTS OF PRIMES

Before describing the general technique, we take two specific illustrations.

(*i*) Consider the pair

$$12600, 660.$$

We express each of these numbers as a product of primes. We have

$$12600 = 2^3 \times 3^2 \times 5^2 \times 7$$

$$660 = 2^2 \times 3 \times 5 \times 11.$$

The prime numbers involved in these two expressions are

$$2, 3, 5, 7, 11.$$

We consider these prime numbers one by one. Of these prime numbers 7, 11 are such that either is a factor of only one of the given numbers so that none of these is a factor of their HCF.

2^2 is the highest power of 2 which is a factor of both the given numbers.
3^1 is the highest power of 3 which is a factor of both the given numbers.
5^1 is the highest power of 5 which is a factor of both the given numbers.
It follows that

$$2^2 \times 3 \times 5$$

is a common factor of the two given numbers. We say that this is the HCF of the given numbers.

In fact if this was not the HCF, the actual HCF would; when expressed as a product of primes, involve a prime factor different from 2, 3, 5 and such a prime factor must appear in the prime factorisation of each of the given numbers.

This, however, is not the case. Thus, we see that the HCF of the given numbers is

$$2^2 \times 3 \times 5 = 60.$$

We now attend to the LCM of the given numbers.

Consider again the expressions of the given numbers as products of primes.

We state that the prime numbers involved are

$$2, 3, 5, 7, 11.$$

We see that the number with prime factorisation

$$2^3 \times 3^2 \times 5^2 \times 7 \times 11$$

is a multiple of both the given numbers *i.e.*, it is a common multiple.

Also it is clear that no smaller power of any of these prime numbers will, when multiplied, be a common multiple.

Working Rule

Let a, b be given numbers We suppose that each of these is expressed as a product of prime factors.

Consider those prime numbers which occur in each of the two prime factorisations.

Then *the product of the smaller powers of the common prime numbers is the HCF. The product of the prime numbers in either or both of these expressions taken with greater powers is the required LCM.*

Illustration. Consider two numbers with the following prime factorisation :

$$a = 2^3 \times 5 \times 11 \times 13^2$$
$$b = 2^2 \times 5^2 \times 11^2 \times 13 \times 17.$$

We have

$$\text{HCF} = 2^2 \times 5 \times 11 \times 13$$
$$\text{LCM} = 2^3 \times 5^2 \times 11^2 \times 13^2 \times 17.$$

EXERCISES

1. Find the HCF of the following sets of numbers by expressing the numbers as products of primes.

(*i*) 594, 5544, 2574
(*ii*) 546, 4095, 4641
(*iii*) 429, 528, 1904
(*iv*) 1230, 14145, 7257
(*v*) 144, 112, 135, 418
(*vi*) 225, 453, 1557, 720
(*vii*) 7, 17, 29, 31, 47
(*viii*) 105, 441, 231, 672, 819
(*ix*) 82, 410, 684, 738, 1026
(*x*) 183, 488, 793, 915, 1220.

2. Find the LCM of the following sets of numbers by expressing the numbers as products of primes.

(*i*) 28, 44, 132
(*ii*) 420, 135, 300
(*iii*) 786, 800, 5168
(*iv*) 105, 252, 360, 700

(*v*) 14, 35, 42, 63, 126 (*vi*) 7, 13, 29, 53, 2

(*vii*) 32, 48, 176, 36, 24 (*viii*) 15, 14, 16, 20, 10

(*ix*) 4, 44, 444, 4444 (*x*) 72, 117, 236, 351.

SUMMARY

The relation 'Is a factor of' in the set of natural numbers.

a is a factor of $b \Leftrightarrow a \mid b \Leftrightarrow b$ is a multiple of a.

$$a \mid b \text{ and } b \mid a \Leftrightarrow a = b$$

$$a \mid b \text{ and } b \mid c \Rightarrow a \mid c$$

$$a \mid b \text{ and } a \mid c \Rightarrow a \mid (b + c)$$

$$a \mid b \text{ and } a \mid c \Rightarrow a \mid (bc).$$

Criteria of divisibility by

2, 3, 4, 5, 6, 8, 9, 10, 11.

HCF and LCM of two and more than two numbers. Algorithm for the determination of the HCF of two numbers.

Product of HCF and LCM of two numbers.

Prime numbers. Composite numbers. Relatively prime pairs of numbers.

Gauss's Theorem :

$$a \mid b\,c \text{ and } a, b \text{ are co-pimes} \Rightarrow a \mid c.$$

Unique prime factorisation theorem.

Computation of the HCF and LCM of sets of numbers through unique prime factorisation.

Every common factor of two numbers is a factor of their highest common factor

Every common multiple of two numbers is a multiple of their lowest common multiple

REVIEW EXERCISES

1. Use Euclid's algorithm to determine which of the following pairs of numbers are co-prime.

(*i*) 385, 931 (*ii*) 3753, 3380

(*iii*) 564, 7963 (*iv*) 17463, 27325.

2. Give five consecutive natural numbers, none of which is a prime.

3. What is the HCF of two consecutive natural numbers ?

4. Show that two consecutive odd numbers are co-prime.

5. Given that a and b are two co-primes, what is the condition that $a + b$ and $a - b$ are also co-primes ?

6. If two natural numbers are squares of natural numbers, show that their HCF and LCM are squares of natural numbers.

7. The HCF of two numbers is 14. What are the two numbers, given that the series of quotients obtained in the division algorithm for determining their HCF is 3, 8, 2, and 4.

8. Show that the square of an odd number other than one diminished by unity is divisible by 8.

9. The LCM of four numbers a, b, c and d is the quotient obtained on dividing the product $abcd$ by the HCF of the four numbers bcd, acd, abd, abc.

10. Obtaian two numbers such that their HCF is 20 and LCM is 420.

11. Obtain two numbers such that their product is 12600 and LCM is 6300

12. Obtain two natural numbers a and b such that

$$a^2 + b^2 = 10530$$

and their LCM is 297.

13. Show that the product $n(n + 1)(n + 2)$ is divisible by 6.

14. Show that the product $n(n + 1)(2n + 1)$ is divisible by 6.

15. Show that the HCF of two numbers does not change if we multiply either of them by a number which is prime relatively to the other.

16. What is the highest power of the prime 7 which divides the product of first five hundred prime numbers.

17. a and b are natural numbers such that

$$a^2 - b^2$$

is a prime number.

Show that

$$a^2 - b^2 = a + b.$$

[Use the property $a^2 - b^2 = (a + b)(a - b)$].

18. If a and b are any two odd primes, show that $a^2 - b^2$ is composite.

19. What is the remainder obtained on dividing the square of an odd natural number by 8 ?

20. What is the remainder obtained on dividing the square of any number by 5 ?

21. Show that number is divisible by 12, if it is divisible by 3 and 4.

22. Show that a number is divisible by 24, if it is divisible by 3 and 8.

23. What are the numbers smaller than 50 and relatively prime to it.

24. If a and b are relatively prime, show that, so are a^2 and b^2.

25. If each of two prime numbers p, q is a factor of a, show that the product pq is also a factor of a.

26. Determine two numbers, knowing their HCF and their sum or their product, as given in the following tables.

I

Sum	72	360	552	420	180	96	168
HCF	9	18	24	12	15	12	24

II

Product	64800	1512	360	2700	840
HCF	18	6	5	6	2

27. If $c \mid a$, $c \mid b$ show that

$$(a + b) \div c = (a \div c) + (b \div c).$$

28. If $c \mid a$, $c \mid b$ show that

$$c \mid (ab).$$

29. A number is said to be perfect, if it is equal to the sum of its factors other than itself. For example, 6 is a perfect number inasmuch as

$$6 = (1 + 2 + 3).$$

There is one other perfect number less than 30. Find that number.

30. A pair of prime numbers is said to be twin primes if they differ by 2 As for example 3, 5 are twin primes.

Give all twin primes less than 100.

3

Fractions

20. INTRODUCTION

In chapter 1, we have seen that whereas it is possible to multiply any two given natural numbers, in that the product of any two natural numbers is a natural number, the position is not so pleasant in respect of the operation of division as the inverse of multiplication. Thus, given any two natural numbers a, b, we cannot in terms of natural numbers, always regard the symbol

$$a \div b$$

as meaningful. In fact the condition for the symbol

$$a \div b$$

to be meaningful in terms of natural numbers is that b is a factor of a. Thus,

$$a \div b \text{ is meaningful} \Leftrightarrow b \mid a.$$

For example, in respect of the set of natural numbers, while the symbol

$$6 \div 3$$

is meaningful, inasmuch as it is the same as 2, the symbol

$$5 \div 3$$

has no meaning.

In this chapter, we propose to invent new numbers. The set of new numbers to be known as the set of *Fractions* will include the set of natural numbers as a sub-set thereof. Also this set of fractions will permit of the possibility of division without any restriction. In fact, it will be seen that every member of this set of fractions will be divisible by every member of the same so that essentially the notion of divisibility in the set of fractions will turn out to be trivial.

In the set of fractions, we shall study the two compositions of addition and multiplication as also the order relation as in the case of the set of natural numbers.

We shall also show that multiplication admits of unrestricted division as the composition, inverse to the same. Of course, subtraction as inverse of addition will still continue to remain a problem in that we shall see that the difference of any two fractions may not be always meaningful. We, however, remark here that a further extension to *rational numbers* in the next chapter will secure relief on this front of subtraction also.

21. NOTION OF A FRACTION

Suppose that there is a bread with ten equal slices and that you take four of them. Then instead of saying that you have four of the ten slices of the bread, one could as well say that you have four-tenths of the bread. There is, however, a third way of making the same statement, *viz.*, that you have

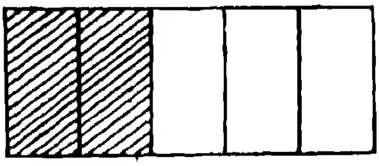
$$4/10 \text{ of the bread,}$$

to be read as 4 by 10 or 4 over 10 of the bread.

In general, suppose that we have any entity, say, a rectangular area, which we divide into b equal parts. Then the part of the total area comprising a of these equal parts may be described as

$$\frac{a}{b} \text{ or } a / b$$

of the area, to be read as a by b or a over b of the area. Here a and b are two natural numbers.



For Example, in respect of the rectangular area in the adjoining diagram, the shaded part is 2/5 of the whole area.

Equality of Parts

We may easily see that what is 4 / 10 of the bread is also 2 / 5 of the bread or 8 / 20 of the bread

In fact, the portion of the bread that we obtain on dividing it into ten slices of equal size and taking four of them is the same as the portion which we obtain on dividing the bread into five equal parts and taking two of those or dividing the bread into twenty equal parts and taking eight out of the twenty.

Again, it may be easily seen that each of the following is the same part of a rupee,

(*i*) 2 / 10 of a rupee (*ii*) 4 / 20 of a rupee,

(*iii*) 1 / 5 of a rupee,

each denoting 20 paise

Further, we see that each of the following portions of a rectangular area is the same area.

(*i*) a / b of the area (*ii*) $2a$ / $2b$ of the area

(*iii*) $3a$ / $3b$ of the area.

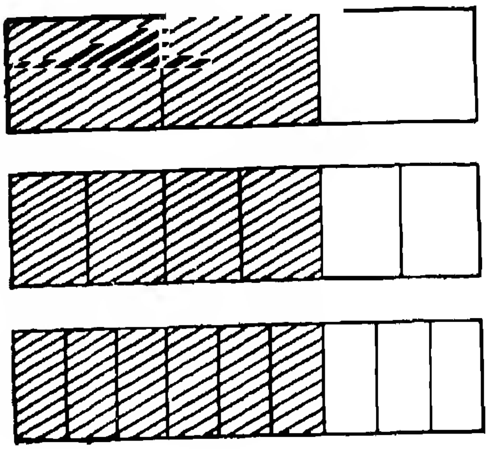
In general,

$$\frac{a}{b} \text{ of an area} = \frac{ak}{bk} \text{ of the area,}$$

where k is any natural number whatsoever.

In fact, if we divide the area into b equal parts and take a of these b parts, the part of the area we shall obtain will be the same as that obtained on taking ak of the bk equal parts.

For example, in the adjoining diagram, the equal shaded portions represent, $2 / 3$ of the area, $4 / 6$ of the area and $6 / 9$ of the area.



It follows, therefore, that a / b of any entity, which is capable of being split up into equal parts, such as length, area, volume, mass, any heap of grain, is the same as ak / bk of the same entity. We can, therefore, multiply a and b by the same natural number without disturbing the part of the entity in question.

It may also be seen that if d is any common factor of a and b, then a / b of the entity is the same as $\frac{a \div d}{b \div d}$ of the same entity. As for example, $6 / 9$ of of the area is the same as $\frac{6 \div 3}{9 \div 3}$ of the area

$$= 2 / 3 \text{ of the area.}$$

In the following, we try to show that

$$\frac{a}{b} \text{ of an entity} = \frac{c}{d} \text{ of the same entity}$$

if $$ad = bc.$$

We have seen that

$$\frac{a}{b} \text{ of an entity} = \frac{ad}{bd} \text{ of the entity}$$

$$= \frac{bc}{bd} \text{ of the entity } (\because ad = bc, \text{ given})$$

$$= \frac{c}{d} \text{ of the entity.}$$

For example, $\frac{4}{6}$ of the area $= \frac{6}{9}$ of the area

in that $4 \times 9 = 6 \times 6.$

Putting Together Two Parts of an Entity

Suppose we have 3/10 of an entity and also 4/10 of the same entity.

Putting together these two portions, we see that the new part, which we obtain, is

$$\frac{3+4}{10} \text{ i.e., } \frac{7}{10} \text{ of the entity}$$

Now suppose that, in general, we have

a/b of an entity,

and also

c/d of the same entity,

the entity being possibly thought of as some rectangular area. We put these two portions together and wish to see a way of describing the new part of the entity which we obtain.

As already seen,

$$\frac{a}{b} \text{ of the entity} = \frac{ad}{bd} \text{ of the entity and}$$

$$\frac{c}{d} \text{ of the entity} = \frac{bc}{bd} \text{ of the entity.}$$

Thus, we see that the two given portions can be described as $\frac{ad}{bd}$ and $\frac{bc}{bd}$ of the entity. Obviously, the two portions put together constitute

$$\frac{ad+bc}{bd} \text{ of the entity.}$$

Part of a Part of an Entity

Let us start with a rupee, consisting as it does of 100 paise.

Consider now

$$\frac{1}{10} \text{ of } \frac{3}{5} \text{ of a rupee.}$$

Now, $\frac{3}{5}$ of a rupee is 60 paise and $\frac{1}{10}$ of 60 paise is 6 paise which is $\frac{6}{100}$ of the rupee. We have, therefore,

$$\frac{1}{10} \text{ of } \frac{3}{5} \text{ of a rupee} = \frac{6}{100} \text{ of a rupee} = \frac{3}{50} \text{ of a rupee.}$$

In general, we take up

$$\frac{a}{b} \text{ of } \frac{c}{d}$$

of a rectangular area.

The problem essentially is that of splitting up c/d of a given rectangular area into b parts and taking a of the same.

Now,

$$\frac{c}{d} \text{ of the area} = \frac{bc}{bd} \text{ of the area}$$

and so in order to take c/d of the area, we may take bc of the bd equal parts in which the given area is supposed to be divided.

Again, we divide $\frac{c}{d}$ of the area into b equal parts so that each part is essentially

$$\frac{c}{bd} \text{ of the area.}$$

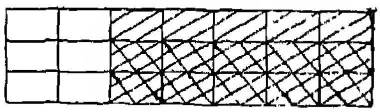
Clearly, a such parts will comprise

$$\frac{ac}{bd} \text{ of the area.}$$

We have, therefore, the following results :

$$\frac{a}{b} \text{ of } \frac{c}{d} \text{ of a rectangular area} = \frac{ac}{bd} \text{ of the same area.}$$

For example, in the adjoining diagram the double shaded area shows



$$\frac{2}{3} \text{ of } \frac{5}{7}$$

of the rectangular area and it is clearly the same as

$$\frac{2 \times 5}{3 \times 7}$$

i e., 10/21 of the area.

Comparison of Parts of an Entity

Suppose that we have

(*i*) $\frac{3}{10}$ of a rupee and (*ii*) $\frac{1}{5}$ of a rupee.

Which of the two is the bigger part of the rupee ?

Now, $\frac{1}{5}$ of a rupee being the same as $\frac{2}{10}$ of a rupee, we see that of the two parts (*i*) and (*ii*), $\frac{3}{10}$ of a rupee is the bigger one.

In general, let us consider

$(i)\frac{a}{b}$ of a rectangular area and $(ii)\frac{c}{d}$ of the same area.

We have,

$\frac{a}{b}$ of the rectangular area is the same as $\frac{ad}{bd}$ of the area

and

$\frac{c}{d}$ of the rectangular area is the same as $\frac{bc}{bd}$ of the area.

We see, therefore, that

$\frac{a}{b}$ of the rectangular area is bigger than $\frac{c}{d}$ of the same

if and only if $ad > bc$.

Meaning of a / b of an Entity when a is Greater Than b

We have so far considered the meaning of a / b of an entity when $a <$ We shall now see what meaning can be given to the concept when $b \leqslant a$.

For the fixation of ideas let us consider 12 / 5 of a bread.

Obviously ten-fifths of a bread means two full breads, so that twelve-fifths of a bread is equivalent to two full breads in addition to two-fifths of a bread. Thus, instead of

$$\frac{12}{5} \text{ cf a bread,}$$

we could as well say

$$2\frac{2}{5} \text{ of a bread.}$$

Again, let us consider a / b of an entity, where $a > b$.

Let q be the quotient and r the remainder obtained on dividing a by b, so that we have

$$a = bq + r \qquad r < b.$$

Here, we are assuming that b is not a factor of a.

Thus, we see that a / b of an entity where

$$a = bq + r$$

is the same as q full entities in addition to r of the b equal parts.

As a particular case, we have

$$\frac{13}{5} \text{ of a rupee} = 2\frac{3}{5} \text{ of a rupee}$$

$$= 2 \text{ rupees and } 60 \text{ paise.}$$

If, however, $a > b$ and b is a factor of a, then there exists a number q such that

$$a = bq$$

and so a / b of an entity will be the same as q full entities.

For example, $\frac{12}{3}$ of a bread is the same as four breads. We may as well note that a / b of an entity denotes the entity itself if $a = b$, as for example, $3 / 3$ of a bread denotes the full bread

Summary of the Results Obtained

Consider an entity which is capable of being split up into any number of equal parts. We denote this entity by E. For the fixation of ideas we may think of E as some length.

We have then the following results.

I. $$\frac{a}{b} \text{ of } E = \frac{c}{d} \text{ of } E$$

if and only if $$ad = bc.$$

II. $\frac{a}{b}$ of E and $\frac{c}{d}$ of E together comprise

$$\frac{ad + bc}{bd} \text{ of } E.$$

III. $$\frac{a}{b} \text{ of } \frac{c}{d} \text{ of } E = \frac{ac}{bd} \text{ of } E.$$

IV. $\frac{a}{b}$ of E is bigger than $\frac{c}{d}$ of E if and only if $ad > bc$.

Remarks. The considerations outlined in this section, referring as they do to parts of entities, enable us to give an abstract definition of the set of fractions. We are also in a position to define the two compositions of addition and multiplication and the relation of order in this new set. The results, we have reached on the basis of a concrete experience, will now be utilized in the form of suggestions to motivate formal definitions.

Thus, we shall now proceed to the world of abstraction on the basis of a concrete experience with parts of entities. The position is similar when we undertake to consider the numbers,

$$1, 2, 3, 4 \ldots\ldots\ldots\ldots$$

instead of the concrete entities,

1 apple, 2 apples, 3 apples, 4 apples, etc.

22. SET OF FRACTIONS

The set of Symbols

$$\frac{a}{b}$$

where a, b are any natural numbers is called the set of **fractions** *which we shall donote by* **F**.

Clearly,

$$\frac{2}{3}, \frac{7}{11}, \frac{13}{25}, \frac{5}{1}, \frac{4}{4}, \frac{13}{3}$$

are all members of the set **F**.

Each member of **F** is called a *fraction*. Thus we have

$$\mathbf{F} = \left\{\frac{a}{b} : a \in \mathbf{N}, b \in \mathbf{N}\right\}.$$

The natural number a is called the **Numerator** and the natural number b is called the **Denominator** of the fraction

$$\frac{a}{b}.$$

We may sometimes write a/b in place of $\frac{a}{b}$.

Equality of Fractions

Let
$$\frac{a}{b}, \frac{c}{d}$$
be two fractions.

We say that these two fractions are equal

if
$$ad = bc$$
and write
$$\frac{a}{b} = \frac{c}{d}$$

Thus, we have

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc.$$

As illustrations, we see that

$$\frac{4}{6} = \frac{6}{9} \text{ for } 4 \times 9 = 6 \times 6.$$

$$\frac{6}{8} = \frac{3}{4} \text{ for } 6 \times 4 = 8 \times 3.$$

We see that if

$$\frac{a}{b} \in \mathrm{F} \text{ and } k \in \mathrm{N},$$

we have

$$\frac{a}{b} = \frac{ak}{bk} \text{ for } a\,(bk) = b\,(ak).$$

Also, if h be any common factor of a and b so that a, b are both divisible by h and $a \div h$, $b \div h$ are natural numbers, we have

$$\frac{a}{b} = \frac{a \div h}{b \div h}.$$

In fact, we have

$$(a \div h)\,h = a, \quad (b \div h)\,h = b.$$

It follows that on multiplying the numerator and denominator of a fraction by the same natural number or on dividing the numerator and denominator by a common factor, we obtain a fraction equal to the given fraction.

As illustrations, we see that

$$\frac{12}{15} = \frac{12 \div 3}{15 \div 3} = \frac{4}{5}$$

$$\frac{x^2y}{xy^2} = \frac{(x^2y) \div (xy)}{(xy^2) \div (xy)} = \frac{x}{y};\ x, y \in \mathrm{N}.$$

Fractions in Lowest Terms

A fraction is said to be in lowest terms if its numerator and denominator have no common factor other than 1 *i.e.*, if they are relatively prime.

Given any fraction

$$\frac{a}{b},$$

there exists a fraction

$$\frac{c}{d}$$

in its lowest terms such that

$$\frac{c}{d} = \frac{a}{b}.$$

In fact, if h be the highest common factor of the natural numbers a, b, we have

$$\frac{a \div h}{b \div h}$$

which is a fraction in its lowest terms equal to the given fraction

$$\frac{a}{b}.$$

Simplification of a Fraction

We say that a fraction c/d is simpler than the fraction a/b

if
$$\frac{c}{d} = \frac{a}{b}$$
and c, d are obtained on dividing a, b by some common factor.

EXERCISES

1. Reduce the following fractions to lowest terms :

(*i*) $\frac{180}{450}$ (*ii*) $\frac{990}{1485}$ (*iii*) $\frac{252}{396}$

(*iv*) 198/462 (*v*) 492/5200 (*vi*) 7360/12144

(*vii*) $\frac{15 \times 42 \times 30}{25 \times 12 \times 42}$ (*viii*) $\frac{84 \times 462}{336 \times 360}$ (*ix*) $\frac{252 \times 342}{420 \times 360}$.

2. Which of the following are true statements ?

(*i*) $\frac{8}{10} \neq \frac{12}{15}$ (*ii*) $\frac{18}{36} = \frac{1}{3}$ (*iii*) $\frac{82}{126} = \frac{2}{3}$.

3. Fill up the blanks with natural numbers to give true statements.

(*i*) $\frac{42}{63} = \frac{2}{-}$ (*ii*) $\frac{-}{54} = \frac{8}{9}$ (*iii*) $\frac{42}{126} = \frac{2}{-}$.

4. Simplify the following, letters denoting natural numbers.

(*i*) $7x/42x$ (*ii*) a^2/a^3 (*iii*) $3a^2b^3/9a^3b^2$

(*iv*) x^2y/xy^2 (*v*) x^5y/x^7 (*vi*) $40ab^2x/120a^3x$

(*vii*) $12acx^2/26a^2c^2x$ (*viii*) $38a^3b^4a^2/57a^4b^4c$ (*ix*) $84a^3m^2n/35a^4mn^3$.

5. Simplify the following, letters denoting natural numbers.

(*i*) $\frac{x + x^2}{y + xy}$ (*ii*) $\frac{2m + m^2}{2m + mn}$ (*iii*) $\frac{2x^2 + 4xy}{3xy + 6y^2}$

(*iv*) $\frac{7 + 14x}{7}$ (*v*) $\frac{ax^2 + a^3}{bx^2 + ba^3}$ (*vi*) $\frac{216x + 450y}{432x + 900y}$

(*vii*) $\frac{216a + 144b + 360c}{576a + 288b + 864c}$ (*viii*) $\frac{35xz + 45yz}{7x + 9y}$.

6. Obtain all the fractions equal to

$$4/11$$

such that the denominator is greater than 300 and smaller than 350.

7. Give the fractions equal to

$$65/117$$

such that the sum of the numerator and denominator is equal to

98, 140, 168.

8. Show from definition that

$$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{c}{d} = \frac{e}{f} \Rightarrow \frac{a}{b} = \frac{e}{f},$$

$\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbf{F}$.

Sum of Fractions

Definition. *Let*

$$\frac{a}{b}, \frac{c}{d}$$

be two fractions. By definition, we have

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and say that the fraction

$$\frac{ad + bc}{bd}$$

is the sum of the fractions

$$\frac{a}{b}, \frac{c}{d}.$$

Note In the first instance, we have to make sure that

$$\frac{a'}{b'}, \frac{c'}{d'}$$

are two fractions such that

$$\frac{a'}{b'} = \frac{a}{b}, \frac{c'}{d'} = \frac{c}{d},$$

then we, as well, have

$$\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} \frac{c'}{d'}.$$

This means that the result of adding the two given fractions is the same as that obtained on adding fractions respectively equal to the given fractions. Let us first consider a particular case.

Consider the fractions

$$\frac{4}{6}, \frac{3}{5}.$$

We have

$$\frac{4}{6} + \frac{3}{5} = \frac{4 \times 5 + 3 \times 6}{6 \times 5} = \frac{38}{30},$$

Also, we have

$$\frac{4}{6} = \frac{2}{3}, \frac{3}{5} = \frac{9}{15},$$

and

$$\frac{2}{3} + \frac{9}{15} = \frac{2 \times 15 + 3 \times 9}{15 \times 3} = \frac{57}{45}.$$

It may easily be seen that

$$\frac{38}{30} = \frac{57}{45}.$$

Let us now consider the general case.

By definition, we have

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a'}{b'} + \frac{c'}{d'} = \frac{a'd' + b'c'}{b'd'}.$$

Also

$$\frac{a'}{b'} = \frac{a}{b} \Rightarrow a'b = ab',$$

$$\frac{c'}{d'} = \frac{c}{d} \Rightarrow c'd = cd'.$$

We have to show that

$$\frac{ad + bc}{bd} = \frac{a'd' + b'c'}{b'd'}.$$

Now

$$\frac{ad + bc}{bd} = \frac{a'd' + b'c'}{b'd'}$$

$$\Leftrightarrow (ad + bc)\, b'd' = (a'd' + b'c')\, bd$$

$$\Leftrightarrow ab'\, dd' + cd'\, bb' = a'b\, dd' + c'd\, bb'.$$

Also

$$ab' = a'b \Rightarrow ab'\, dd' = a'b\, dd'$$

$$cd' = c'd \Rightarrow cd'\, bb' = c'd\, bb'$$

and these jointly imply that

$$ab'\, dd' + cd'\, bb' = a'b\, dd' + c'd\, bb'.$$

Thus, we see that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} = \frac{a'd' + b'c'}{b'd'} = \frac{a'}{b'} + \frac{c'}{d'}.$$

EXERCISES

1. Compute the following sums.

(*i*) $\frac{2}{3} + \frac{4}{5}$ (*ii*) $\frac{4}{5} + \frac{2}{3}$

(*iii*) $\frac{7}{8} + \frac{8}{9}$ (*iv*) $\frac{8}{9} + \frac{7}{8}$.

2. Compute the following sums.

(i) $\frac{4}{7} + \left(\frac{13}{12} + \frac{9}{5}\right)$ (ii) $\left(\frac{4}{7} + \frac{13}{12}\right) + \frac{9}{5}$

(iii) $\frac{1}{2} + \left(\frac{2}{3} + \frac{3}{4}\right)$ (iv) $\left(\frac{1}{2} + \frac{2}{3}\right) + \frac{3}{4}$.

3. Show that

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}.$$

$$\left[\text{We have } \frac{a}{b} + \frac{c}{b} = \frac{ab+cb}{bb} = \frac{(a+c)\,b}{bb} = \frac{a+c}{b}.\right]$$

Properties of Addition Composition in the Set of Fractions

We proceed to show that the addition composition in **F** is both commutative and associative.

Theorem. *Addition composition in the set of fractions is commutative, i.e.,*

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b} \qquad \forall\ \frac{a}{b}, \frac{c}{d} \in \mathbf{F}.$$

Proof.

We have

$$\frac{a}{b} = \frac{ad}{bd},$$

$$\frac{c}{d} = \frac{bc}{bd},$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} = \frac{bc+ad}{bd}$$

$$= \frac{bc}{bd} + \frac{ad}{bd} = \frac{c}{d} + \frac{a}{b}.$$

It will be seen that to prove that the addition in **F** is commutative, we have made use of the fact that addition in **N** is commutative.

Theorem. *Addition composition in the set of fractions is associative, i.e.,*

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) \ \forall\ \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbf{F}.$$

Proof.

We have $\qquad \frac{a}{b} = \frac{adf}{bdf}, \frac{c}{d} = \frac{cbf}{dbf}, \frac{e}{f} = \frac{ebd}{fbd}.$

Also we have

$$dbf = bdf = fbd\ ;\ b, d, f \in \mathbf{N}.$$

Thus,

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{ad+bc}{bd}+\frac{e}{f}$$

$$=\frac{adf+bcf}{bdf}+\frac{ebd}{bdf}$$

$$=\frac{(adf+bcf)+ebd}{bdf}$$

$$=\frac{adf+(bcf)+ebd}{bdf}$$

$$=\frac{adf}{bdf}+\frac{bcf+ebd}{bdf}$$

$$=\frac{a}{b}+\frac{cf+ed}{df}$$

$$=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right).$$

It will be seen that in the above process, we have employed the Commutative, Associative and Distributive properties of multiplication in the set of natural numbers.

Note. As in the set of natural numbers, the addition in the set of fractions also satisfies the cancellation law. This cancellation law in the set **F** will be stated and proved after we have defined the relation 'Is greater than' in this set.

Product of Fractions. Let

$$\frac{a}{b}, \frac{c}{d} \in \mathbf{F}.$$

Definition. We write

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

and call the fraction

$$\frac{ac}{bd}$$

as the product of the fractions

$$\frac{a}{b}, \frac{c}{d}.$$

We may also write

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

or simply
$$\frac{a}{b} \quad \frac{a}{d} = \frac{ac}{bd}.$$

Note. As in the case of the addition composition in F, we shall also, in respect of the multiplication composition in F, show that
$$\frac{a'}{b'} = \frac{a}{b} \text{ and } \frac{c'}{d'} = \frac{c}{d} \Rightarrow \frac{a'}{b'} \cdot \frac{c'}{d'} = \frac{a}{b} \cdot \frac{c}{d}$$
so that the product of fractions is not changed when we replace the same by fractions equal to them

Now
$$\frac{a'}{b'} = \frac{a}{b} \Leftrightarrow a'b = ab'$$
$$\frac{c'}{d'} = \frac{c}{d} \Leftrightarrow c'd = cd'$$
and these imply
$$(a'b)\,(c'd) = (ab')\,(cd').$$
Again
$$\frac{a'}{b'} \quad \frac{c'}{d'} = \frac{a'c'}{b'd'},\ \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
and
$$\frac{a'c'}{b'd'} = \frac{ac}{bd} \Leftrightarrow (a'c')\,(bd) = (ac)\,(b'd')$$
$$\Leftrightarrow (a'b)\,(c'd) = (ab')\,(cd').$$
Thus, we see that
$$\frac{a'}{b'} \cdot \frac{c'}{d'} + \frac{a'c'}{b'd'} = \frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d}.$$

EXERCISES

1. Compute the following products of fractions.

(i) $\frac{2}{3} \times \frac{4}{5}$ (ii) $\frac{4}{5} \times \frac{2}{3}$

(iii) $\frac{7}{8} \times \frac{11}{13}$: (iv) $\frac{11}{13} \times \frac{7}{8}$

(v) $\frac{3}{4} \times \left(\frac{5}{7} \times \frac{9}{11}\right)$ (vi) $\left(\frac{3}{4} \times \frac{5}{7}\right) \times \frac{9}{11}$

(vii) $\frac{12}{13} \times \left(\frac{3}{4} \times \frac{2}{7}\right)$ (viii) $\left(\frac{12}{13} \times \frac{3}{4}\right) \times \frac{2}{7}$.

2. Compute the following.

(i) $\frac{2}{3} \times \left(\frac{3}{4} + \frac{1}{2}\right)$ (ii) $\left(\frac{2}{3} \times \frac{3}{4}\right) + \left(\frac{2}{3} \times \frac{1}{2}\right)$

(iii) $\frac{5}{7} \times \left(\frac{7}{8} + \frac{8}{9}\right)$ (iv) $\left(\frac{5}{7} \times \frac{7}{8}\right) + \left(\frac{5}{7} \times \frac{8}{9}\right)$.

3. Compute and simplify the following products of fractions, letters denoting natural numbers.

(i) $\frac{2a}{3} \times \frac{3b}{4}$ (ii) $\frac{a^2}{b^2} \times \frac{b}{a}$

(iii) $\frac{3x}{16} \times \frac{4}{y}$ (iv) $\frac{3a^2b}{4} \times \frac{4b^2}{3}$

(v) $\frac{5x^2y^2}{8} \times \frac{8xy}{3}$ (vi) $\frac{9x^2y}{4} \times \frac{2y^3z}{9}$

(vii) $\frac{3ab^2}{4} \times \left(\frac{4b^2c}{5} \times \frac{5ac}{3}\right)$ (viii) $\left(\frac{3ab^2}{4} \times \frac{4b^2c}{5}\right) \times \frac{5ac}{3}$

(ix) $\frac{15xyz}{27xy} \times \left(\frac{9x^2y}{5x^3} \times \frac{17xyz^2}{9}\right)$ (x) $\left(\frac{15xyz}{27xy} \times \frac{9x^2y}{5x^3}\right) \times \frac{17xyz^2}{9}$

(xi) $\frac{a^2b^2}{xy} \times \left(\frac{xy}{1} \times \frac{7abx}{ax}\right)$.

4. Show that

(i) $\frac{a}{b} \times \frac{1}{1} = \frac{a}{b}\ \forall\ \frac{a}{b} \in \mathbf{F}$

(ii) $\frac{a}{b} \times \frac{b}{a} = \frac{1}{1}\ \forall\ \frac{a}{b} \in \mathbf{F}$.

Properties of the Multiplication Composition in **F**.

Theorem. *Multiplication Composition in* **F** *is commutative.*

Let

$$\frac{a}{b}, \frac{c}{d} \in \mathbf{F}.$$

We have

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

$$= \frac{ca}{db} = \frac{c}{d} \times \frac{a}{b}.$$

Thus,

$$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b} \qquad \forall\ \frac{a}{b}, \frac{c}{d} \in \mathbf{F}.$$

Hence the result.

Theorem. *Multiplication composition in the set of fractions is associative.*

Proof. Let

$$\frac{a}{b}, \frac{c}{d}, \frac{e}{} \in \mathbf{F}.$$

We have

$$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f}$$
$$= \frac{(ac)\,e}{(bd)\,f}$$
$$= \frac{a(ce)}{b(df)}$$
$$= \frac{a}{b} \times \frac{ce}{df}$$
$$= \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)$$

so that

$$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)$$

$$\forall\ \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbf{F}.$$

Hence the result.

Multiplicative Property of the Fraction $\frac{1}{1}$.

Theorem. We have

$$\forall\ \frac{a}{b} \in \mathbf{F}.$$

$$\frac{a}{b} \times \frac{1}{1} = \frac{a \times 1}{b \times 1} = \frac{a}{b}.$$

Note. Because of the property proved above, the fraction $\frac{1}{1}$ is referred to as the multiplicative identity. We shall call it the unity.

Reciprocal of a Fraction.

Theorem. *To each fraction*

$$\frac{a}{b}$$

there corresponds the fraction

$$\frac{b}{a},$$

such that

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{1}{1}.$$

Because of this property, we say that $\frac{b}{a}$ is the multiplicative inverse of the fraction $\frac{a}{b}$. We shall also refer to $\frac{b}{a}$ as the reciprocal of $\frac{a}{b}$. Of course $\frac{a}{b}$ is as well the reciprocal of $\frac{b}{a}$.

Thus, of the two fractions, each is the reciprocal of the other if the numerator of one is the denominator of the other and *vice versa*. As illustrations, we see that

$$\text{the reciprocal of } \frac{7}{11} \text{ is } \frac{11}{7},$$

$$\text{the reciprocal of } \frac{2}{1} \text{ is } \frac{1}{2}.$$

Division in F.

We are now in a position to vindicate the stand we took in respect of the need for the study of fractions and show that given any two fractions

$$\frac{a}{b}, \frac{c}{d},$$

there exists a fraction

$$\frac{e}{f}$$

such that

$$\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d}.$$

Clearly

$$\frac{e}{f} = \frac{bc}{ad}$$

will do the job inasmuch as we have

$$\begin{aligned} \frac{a}{b} \frac{bc}{ad} &= \frac{abc}{bad} \\ &= \frac{(abc) \div ab}{(abd) \div ab} = \frac{c}{d}. \end{aligned}$$

We write

$$\frac{e}{f} = \frac{c}{d} \div \frac{a}{b}$$

and say that $\frac{e}{f}$ is obtained on dividing $\frac{c}{d}$ by $\frac{a}{b}$. Clearly we have

$$\frac{c}{d} \div \frac{a}{b} = \frac{bc}{ad}.$$

We have

$$\frac{c}{d} \div \frac{a}{b} = \frac{cb}{da} = \frac{c}{d} \quad \frac{b}{a}$$

so that, to divide $\frac{c}{d}$ by $\frac{a}{b}$, we multiply $\frac{c}{d}$ with the reciprocal $\frac{b}{a}$ of $\frac{a}{b}$.

As an illustration, we see that

$$\frac{2}{3} \div \frac{3}{5} = \frac{2}{3} \cdot \frac{5}{3} = \frac{10}{9}$$

$$\frac{7}{8} \div \frac{13}{12} = \frac{7}{8} \cdot \frac{12}{13} = \frac{84}{104} = \frac{21}{26}.$$

We may sometimes write

$$\frac{c}{d}\Big/\frac{a}{b} \text{ or } \frac{\frac{c}{d}}{\frac{a}{b}}$$

in place of

$$\frac{c}{d} \div \frac{a}{b}.$$

It will thus be seen that in respect of the set F of fractions, each member of F is divisible by every member thereof. The statement, however, does not remain true if we replace F by N.

EXERCISES

1. Give the reciprocals of the following fractions, letters denoting natural numbers.

(i) $\frac{7}{11}$ (ii) $\frac{12}{17}$ (iii) $\frac{22}{27}$

(iv) $\frac{2a}{3}$ (v) $\frac{7a}{6b}$ (vi) $\frac{2}{3a}$

(vii) $\frac{a}{1}$ (viii) $\frac{1}{b}$ (ix) $\frac{a^2}{b^2}$.

2. Simplify the following, letters denoting natural numbers.

(i) $\frac{2a^2b^3}{3a^3b^2} \div \frac{5ab^4}{7a^4b^3}$ (ii) $\frac{7mn^2}{5m^3n} \div \frac{3mn}{8m^2n^4}$

(iii) $\frac{4x^2y}{5a^2b} \div \frac{2xy^2}{15ab^2}$ (iv) $\frac{35ab}{8} \div \frac{7a}{2}$

(v) $\frac{5a}{7} \div \frac{2a}{3}$ (vi) $\frac{4x}{y} \div \frac{3y}{x}$.

Distributive Law. *Multiplication distributes addition in the set of fractions.* We shall prove that

$$\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\ \frac{e}{f}$$

$\forall\ \frac{a}{b},\ \frac{c}{d},\ \frac{e}{f}\in \mathbf{F}$

Proof. We have

$$\frac{a}{b}\cdot\left(c+\frac{e}{f}\right)=\frac{a}{b}\cdot\frac{cf+ed}{df}$$

$$=\frac{a\,(cf+ed)}{bdf}$$

$$=\frac{acf+aed}{bdf}$$

$$=\frac{acf}{bdf}+\frac{aed}{bdf}$$

$$=\frac{ac}{bd}+\frac{ae}{bf}$$

$$=\frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\cdot\frac{e}{f}.$$

EXERCISE

Simplify the following in two ways, letters denoting natural numbers.

(*i*) $\frac{2a}{3b}\left(\frac{a}{6b}+\frac{b}{a}\right)$ (*ii*) $\frac{ab}{1}\left(\frac{1}{a}+\frac{1}{b}\right)$

(*iii*) $\frac{ab}{a^2+b^2}\left(\frac{a}{b}+\frac{b}{a}\right)$ (*iv*) $\left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{a}{b}+\frac{c}{d}\right)$.

23. ORDER RELATION IN THE SET OF FRACTIONS

Let

$$\frac{a}{b},\ \frac{c}{d}$$

be two fractions.

We have already seen that

$$\frac{a}{b}=\frac{c}{d}\Leftrightarrow ad=bc.$$

We now define the relation

'Is greater than'

in the set F of fractions.

Definition. *We say that*

$$\frac{a}{b}\ \textit{is greater than}\ \frac{c}{d}$$

if $$ad > bc$$
and, in symbols, write

$$\frac{a}{b} > \frac{c}{d}.$$

Thus, by definition,

$$\frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc.$$

Also if $\frac{a}{b}$ is greater than $\frac{c}{d}$ we say that $\frac{c}{d}$ s smaller than $\frac{a}{b}$ and write

$$\frac{c}{d} < \frac{a}{b}.$$

Thus, we have

$$\frac{c}{d} < \frac{a}{b} \Leftrightarrow \frac{a}{b} > \frac{c}{d}$$

The symbols, $\geqslant, \leqslant$.

If $\frac{a}{b}, \frac{c}{d}$ be two fractions such that

$$\frac{a}{b} > \frac{c}{d} \text{ or } \frac{a}{b} = \frac{c}{d},$$

we write

$$\frac{a}{b} \geqslant \frac{c}{d}$$

and read it as follows :

$\frac{a}{b}$ *is greater than or equal to* $\frac{c}{d}$.

Similarly, we have

$$\frac{a}{b} \leqslant \frac{c}{d} \Leftrightarrow ad \leqslant bc.$$

As illustrations, we have

(*i*) $\frac{4}{5} > \frac{2}{3}$ (*ii*) $\frac{11}{13} > \frac{2}{3}$

(*iii*) $\frac{5}{7} > \frac{3}{7}$.

It needs to be shown that

if $$\frac{a}{b} = \frac{a'}{b'}, \quad \frac{c}{d} = \frac{c'}{d'},$$

then

$$\frac{a}{b} > \frac{c}{d} \Leftrightarrow \frac{a'}{b'} > \frac{c'}{d'}.$$

We have

$$\frac{a}{b} = \frac{a'}{b'} \Leftrightarrow ab' = a'b\,; \frac{c}{d} = \frac{c'}{d'} \Leftrightarrow cd' = c'd.$$

Now

$$\frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc.$$

We have to show that

$$a'd' > b'c'.$$

For this we prove that $a'd' \neq b'c'$ and $a'd' \not< b'c'$.

Now $\quad ad > bc,\ a'd' = b'c' \Rightarrow (ad)\,(b'c') > (bc)\,(a'd')$

$\quad\quad ad > bc,\ b'c' > a'd' \Rightarrow (ad)\,(b'c') > (bc)\,(a'd').$

Also $\quad ab' = a'b,\ cd' = c'd \Rightarrow (ad)\,(b'c') = (bc)\,(a'd')$

It follows that $\quad a'd' > b'c'$

which is equivalent to $\quad \dfrac{a'}{b'} > \dfrac{c'}{d'}.$

EXERCISES

Replace the sign ? by one of the appropriate signs $>$, $<$, $=$

1.

(i) $\dfrac{4}{5}$? $\dfrac{2}{3}$ (ii) $\dfrac{7}{8}$? $\dfrac{9}{11}$ (iii) $\dfrac{3}{4}$? $\dfrac{4}{5}$

(iv) $\dfrac{22}{36}$? $\dfrac{33}{54}$ (v) $\dfrac{9}{11}$? $\dfrac{7}{8}$ (vi) $\dfrac{14}{30}$? $\dfrac{21}{45}$.

2.

(i) $\dfrac{3}{4}$? $\dfrac{7}{8}$ (ii) $\dfrac{3}{4}\cdot\dfrac{5}{6}$? $\dfrac{7}{8}\cdot\dfrac{5}{6}$

(iii) $\dfrac{5}{11}$? $\dfrac{2}{7}$ (iv) $\dfrac{5}{11}\cdot\dfrac{7}{16}$? $\dfrac{2}{7}\cdot\dfrac{7}{16}$

(v) $\dfrac{13}{15}$? $\dfrac{17}{21}$ (vi) $\dfrac{13}{15}\cdot\dfrac{7}{8}$? $\dfrac{17}{21}\cdot\dfrac{7}{8}$.

3.

(i) $\dfrac{3}{4}$? $\dfrac{5}{8}$ (ii) $\dfrac{3}{7}\cdot\dfrac{5}{6}$? $\dfrac{7}{8}\cdot\dfrac{5}{9}$

(iii) $\dfrac{5}{13}$? $\dfrac{3}{7}$ (iv) $\dfrac{5}{13}\times\dfrac{7}{17}$? $\dfrac{2}{9}\times\dfrac{7}{15}$

(v) $\frac{13}{18} ? \frac{19}{21}$ (vi) $\frac{13}{18} \cdot \frac{5}{8} ? \frac{17}{23} \cdot \frac{7}{9}$.

4.

(i) $\frac{2}{3} ? \frac{3}{4}$ (ii) $\frac{2}{3} + \frac{5}{6} ? \frac{3}{4} + \frac{5}{6}$

(iii) $\frac{7}{11} ? \frac{3}{5}$ (iv) $\frac{7}{11} + \frac{5}{7} ? \frac{3}{5} + \frac{5}{7}$

(v) $\frac{12}{13} ? \frac{14}{15}$ (vi) $\frac{12}{13} + \frac{8}{9} ? \frac{14}{15} + \frac{8}{9}$.

5.

(i) $\frac{2}{3} ? \frac{5}{7}$ (ii) $\frac{2}{3} ? \frac{1}{2}\left(\frac{2}{3} + \frac{5}{7}\right) ? \frac{5}{7}$

(iii) $\frac{12}{17} ? \frac{18}{23}$ (iv) $\frac{12}{17} ? \frac{1}{2}\left(\frac{12}{17} + \frac{18}{23}\right) ? \frac{18}{23}$

(v) $\frac{21}{35} ? \frac{17}{19}$ (vi) $\frac{12}{35} ? \frac{1}{2}\left(\frac{21}{35} + \frac{17}{19}\right) ? \frac{17}{19}$.

6. Arrange the following finite sets of fractions in ascending and descending orders. (A finite set of fractions is said to be arranged in an ascending order if the arrangement is such that every fraction is followed by one greater than it. We have a similar definition for arrangement in descending order.)

(i) $\left\{\frac{1}{2}, \frac{1}{7}, \frac{1}{3}, \frac{1}{11}, \frac{1}{13}, \frac{1}{4}, \frac{1}{8}\right\}$

(ii) $\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{11}{12}, \frac{5}{6}, \frac{14}{15}\right\}$

(iii) $\left\{\frac{3}{1}, \frac{7}{1}, \frac{6}{1}, \frac{8}{1}, \frac{4}{2}\right\}$.

Properties of the Order Relation *'Is greater than'.*

Trichotomy Law. *Given any two fractions*

$$\frac{a}{b}, \frac{c}{d}$$

there exists one and only one of the three possibilities

(i) $\frac{a}{b} = \frac{c}{d}$ (ii) $\frac{a}{b} > \frac{c}{d}$ (iii) $\frac{c}{d} > \frac{a}{b}$.

Proof. We have

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$$

$$\frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc$$

$$\frac{c}{d} > \frac{a}{b} \Leftrightarrow bc > ad.$$

Also a, b, c, d being natural numbers, we have one and only one of the three possibilities.

(*i*) $ad = bc$ (*ii*) $ad > bc$ (*iii*) $bc > ad$.

Hence the result.

Transitivity of the Relation

Theorem. $\frac{a}{b} > \frac{c}{d}$ and $\frac{c}{d} > \frac{e}{f} \Rightarrow \frac{a}{b} > \frac{e}{f}$.

Proof. We have

$$\frac{a}{b} = \frac{adf}{bdf}, \frac{c}{d} = \frac{cbf}{dbf}, \frac{e}{f} = \frac{ebd}{fbd}.$$

Now

$$\frac{a}{b} > \frac{c}{d} \Rightarrow \frac{adf}{bdf} > \frac{cbf}{dbf} \Rightarrow adf > cbf$$

$$\frac{c}{d} > \frac{e}{f} \Rightarrow \frac{cbf}{dbf} > \frac{ebd}{fbd} \Rightarrow cbf > ebd.$$

Again

$$adf > cbf \text{ and } cbf > ebd \Rightarrow adf > ebd$$

and

$$adf > ebd \Rightarrow af > be \Rightarrow \frac{a}{b} > \frac{e}{f}.$$

Hence the result.

Compatibility of the Relation 'Is Greater Than' with Addition Composition. Cancellation Law for Addition.

Theorem. $\frac{a}{b} > \frac{c}{d} \Leftrightarrow \frac{a}{b} + \frac{e}{f} > \frac{c}{d} + \frac{e}{f}$.

Firstly we prove that

$$\frac{a}{b} > \frac{c}{d} \Rightarrow \frac{a}{b} + \frac{e}{f} > \frac{c}{d} + \frac{e}{f}.$$

Proof. We have

$$\frac{a}{b} = \frac{adf}{bdf}, \frac{c}{d} = \frac{cbf}{dbf}, \frac{e}{f} = \frac{ebd}{fbd}.$$

Now

$$\begin{aligned} \frac{a}{b} > \frac{c}{d} &\Rightarrow \frac{adf}{bdf} > \frac{cbf}{dbf} \\ &\Rightarrow adf > cbf \\ &\Rightarrow adf + ebd > cbf + ebd \\ &\Rightarrow \frac{adf + ebd}{bdf} > \frac{cbf + ebd}{bdf} \\ &\Rightarrow \frac{adf}{bdf} + \frac{ebd}{bdf} > \frac{cbf}{bdf} + \frac{ebd}{bdf} \\ &\Rightarrow \frac{a}{b} + \frac{e}{f} > \frac{c}{d} + \frac{e}{f}. \end{aligned}$$

Conversely, we prove that

$$\frac{a}{b}+\frac{e}{f}>\frac{c}{d}+\frac{e}{f}\Rightarrow\frac{a}{b}>\frac{c}{d}.$$

Proof. We are given that

$$\frac{a}{b}+\frac{e}{f}>\frac{c}{d}+\frac{e}{f}.$$

We have

$$\frac{a}{b}=\frac{c}{d}\Rightarrow\frac{a}{b}+\frac{e}{f}=\frac{c}{d}+\frac{e}{f}$$

$$\frac{c}{d}>\frac{a}{b}\Rightarrow\frac{c}{d}+\frac{e}{f}>\frac{a}{b}+\frac{e}{f}.$$

It now follows by the Trichotomy Law that

$$\frac{a}{b}+\frac{e}{f}>\frac{c}{d}+\frac{e}{f}\Rightarrow\frac{a}{b}>\frac{c}{d},$$

Hence the result.

Cor. $$\frac{a}{b}=\frac{c}{d}\Leftrightarrow\frac{a}{b}+\frac{e}{f}=\frac{c}{d}+\frac{e}{f}.$$

Compatibility of the Relation 'Is Greater Than' with Multiplication Composition. Cancellation Law for Multiplication.

Theorem.

$$\frac{a}{b}>\frac{c}{d}\Leftrightarrow\frac{a}{b}\cdot\frac{e}{f}>\frac{c}{d}\cdot\frac{e}{f}.$$

Firstly we prove that

$$\frac{a}{b}>\frac{c}{d}\Rightarrow\frac{a}{b}\cdot\frac{e}{f}>\frac{c}{d}\cdot\frac{e}{f}.$$

Proof.

$$\begin{aligned}\frac{a}{b}>\frac{c}{d}&\Rightarrow ad>bc\\&\Rightarrow (ad)\,(ef)\triangleright(bc)\,(ef)\\&\Rightarrow (ad)\,(ef)\triangleright(ce)\,(bf)\\&\Rightarrow (ae)\,(df)\triangleright(ce)\,(bf)\\&\Rightarrow\frac{ae}{bf}>\frac{ce}{df}\\&\Rightarrow\frac{a}{b}\cdot\frac{e}{f}>\frac{c}{d}\cdot\frac{e}{f}.\end{aligned}$$

We now prove the converse, *i.e.*,

$$\frac{a}{b}\quad\frac{e}{f}>\frac{c}{d}\cdot\frac{e}{f}\Rightarrow\frac{a}{b}\cdot\frac{c}{d}.$$

Proof. We are given that

$$\frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f}.$$

Now

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}$$

$$\frac{c}{d} > \frac{a}{b} \Rightarrow \frac{c}{d} \cdot \frac{e}{f} > \frac{a}{b} \cdot \frac{e}{f}$$

$$\Leftrightarrow \frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f}.$$

It now follows that

$$\frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f} \Rightarrow \frac{a}{b} > \frac{c}{d}.$$

Cor.

$$\frac{c}{b} = \frac{c}{d} \Leftrightarrow \frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}.$$

The result could also be proved without use of the order relation. In fact, we have

$$\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f} \Leftrightarrow \left(\frac{a}{b} \cdot \frac{e}{f}\right)\frac{f}{e} = \left(\frac{c}{d} \cdot \frac{e}{f}\right)\frac{f}{e}$$

$$\Leftrightarrow \frac{a}{b}\left(\frac{e}{f} \cdot \frac{f}{e}\right) = \frac{c}{d}\left(\frac{e}{f} \cdot \frac{f}{e}\right)$$

$$\Leftrightarrow \frac{a}{b} \cdot \frac{1}{1} = \frac{c}{d} \cdot \frac{1}{1}$$

$$\Leftrightarrow \frac{a}{b} = \frac{c}{d}.$$

24. SUBTRACTION

Theorem. *Given two fractions*

$$\frac{a}{b}, \frac{c}{d}$$

such that

$$\frac{a}{b} > \frac{c}{d}$$

there exists one and only one fraction

$$\frac{e}{f}$$

such that

$$\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$$

Proof. Now

$$\frac{a}{b} > \frac{c}{d} \Rightarrow ad > bc.$$

Again

$$ad > bc \Rightarrow \exists\ x \in \mathbf{N} \text{ such that}$$

$$ad = bc + x$$

$$\Rightarrow \frac{ad}{bd} = \frac{bc + x}{bd}$$

$$\Rightarrow \frac{ad}{bd} = \frac{bc}{bd} + \frac{x}{bd}$$

$$\Rightarrow \frac{a}{b} = \frac{c}{d} + \frac{x}{bd}$$

so that $x/(bd)$ is the fraction required.

Surely $x = ad - bc$ and so the required fraction is

$$\frac{ad - bc}{bd}$$

We now show the uniqueness.

Let, if possible,

$$\frac{e}{f}, \frac{e'}{f'}$$

be two fractions such that

$$\frac{c}{d} + \frac{e}{f} = \frac{a}{b} = \frac{c}{d} + \frac{e'}{f'}.$$

Then

$$\frac{c}{d} + \frac{e}{f} = \frac{c}{d} + \frac{e'}{f'} \Rightarrow \frac{e}{f} = \frac{e'}{f'}.$$

This proves the uniqueness part.

Definition. *We write*

$$\frac{e}{f} = \frac{a}{b} - \frac{c}{d}.$$

Thus, $$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

We see that

$$\frac{a}{b} - \frac{c}{d} = \frac{e}{f} \Leftrightarrow \frac{a}{b} = \frac{c}{d} + \frac{e}{f}.$$

It would be seen that

$$\frac{a}{b} - \frac{c}{d}$$

is meaningful if and only if

$$\frac{a}{b} > \frac{c}{d}$$

Theorem. If $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$

are three fractions such that

$$\frac{a}{b} > \left(\frac{c}{d} + \frac{e}{f}\right)$$

then
$$\frac{a}{b} - \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} - \frac{c}{d}\right) - \frac{e}{f}.$$

Proof. There exists a fraction g/h such that

$$\frac{a}{b} = \left(\frac{c}{d} + \frac{e}{f}\right) + \frac{g}{h} = \frac{c}{d} + \left(\frac{e}{f} + \frac{g}{h}\right)$$

$$\Rightarrow \frac{a}{b} - \frac{c}{d} = \frac{e}{f} + \frac{g}{h}$$

$$\Rightarrow \left(\frac{a}{b} - \frac{c}{d}\right) - \frac{e}{f} = \frac{g}{h} = \frac{a}{b} - \left(\frac{c}{d} + \frac{e}{f}\right).$$

Thus, we get

$$\frac{a}{b} - \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} - \frac{c}{d}\right) - \frac{e}{f}.$$

Note
$$\frac{a}{b} > \left(\frac{c}{d} + \frac{e}{f}\right)$$

$$\Rightarrow \frac{a}{b} > \frac{c}{d} \text{ and } \frac{a}{b} > \frac{e}{f}.$$

25. DECIMAL FRACTIONS

Definition. *A fraction*

$$\frac{a}{b}$$

is said to be a **decimal fraction,** *if it is equal to a fraction whose denominator is some power of* 10, *i e.,*

$$10^1, 10^2, 10^3 \textit{ etc.}$$

For example,

$$\frac{3}{10}, \frac{27}{100}, \frac{31}{1000}$$

are decimal fractions.

Again each of the fractions

$$\frac{1}{2}, \frac{3}{5}, \frac{7}{25}$$

is a decimal fraction in that we have

$$\frac{1}{2} = \frac{5}{10}$$

$$\frac{3}{5} = \frac{6}{10}$$

$$\frac{7}{25} = \frac{28}{100}.$$

It follows that a fraction is a decimal fraction if, when expressed in its lowest terms the only prime factors appearing in the prime factorisation of the denominator are 2 and (or) 5.

For example, the fraction

$$\frac{7}{2^4}$$

whose denominator is a power of 2 is a decimal fraction in that we have

$$\frac{7}{2^4} = \frac{7 \times 5^4}{2^4 \times 5^4} = \frac{7 \times 5^4}{10^4}.$$

Again

$$\frac{3}{5^3}$$

is a decimal fraction in that we have

$$\frac{3}{5^3} = \frac{3 \times 2^3}{5^3 \times 2^3} = \frac{3 \times 2^3}{10^3}.$$

EXERCISE

Which of the following fractions are decimal fractions ?

(*i*) $\frac{1}{2}$ (*ii*) $\frac{21}{75}$ (*iii*) $\frac{6}{14}$

(*iv*) $\frac{1}{15}$ (*v*) $\frac{3}{50}$ (*vi*) $\frac{7}{20}$.

A Notation for Decimal Fractions (*Decimal Notation*). Consider a decimal fraction

27/100.

We have

$$\frac{27}{100} = \frac{2 \times 10 + 7}{100} = \frac{2 \times 10}{100} + \frac{7}{100}$$

$$= \frac{2}{10} + \frac{7}{10^2}$$

which we shall write as

$$\cdot 27.$$

Again consider $\frac{3}{4}$, we have

$$\frac{3}{4} = \frac{3 \times 25}{4 \times 25} = \frac{75}{100} = \frac{7 \times 10 + 5}{100}$$

$$= \frac{7}{10} + \frac{5}{10^2} = \cdot 75.$$

We now consider the converse of the above situation. In terms of what has been stated above, we have

$$37\cdot 234 = 3 \times 10 + 7 + \frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3}$$

$$= 37 + \frac{200}{10^3} + \frac{30}{10^3} + \frac{4}{10^3}$$

$$= 37 + \frac{200 + 30 + 4}{10^3} = 37\frac{234}{10^3}$$

$$= 37\frac{234}{1000} = \frac{37234}{1000}.$$

EXERCISES

1. Express the following in decimal notation.

(*i*) $\frac{13}{20}$ (*ii*) $\frac{15}{32}$ (*iii*) $\frac{7}{125}$

(*iv*) $\frac{19}{250}$ (*v*) $\frac{17}{320}$ (*vi*) $\frac{217}{160}$.

2. Express the following in the form a/b.

(*i*) $\cdot 324$ (*ii*) $2\cdot 0123$ (*iii*) $27\cdot 45$

(*iv*) $2\cdot 123$ (*v*) $\cdot 1357$ (*vi*) $31\cdot 1234$.

Theorem. *The sum and product of two decimal fractions is a decimal fraction.*

Proof. Consider two decimal fractions

$$\frac{a}{b}, \frac{c}{d}.$$

suppose that they are in their lowest terms. We have

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}.$$

Since b, d have 2 and 5 only as prime factors, we see that their product can have no prime factor other than 2 and 5.

Again we have

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

so that, as in the case of the sum,

$$\frac{ac}{bd}$$

is a decimal fraction

Case of Difference. Consider two decimal fractions

$$\frac{a}{b}, \frac{c}{d}$$

such that

$$\frac{a}{b} > \frac{c}{d}.$$

In this case the difference

$$\frac{a}{b} - \frac{c}{d}$$

is a fraction and we have

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$$

so that as before, we see that the difference of two decimal fractions, if it exists, is also a decimal fraction.

Case of Division. Let

$$\frac{a}{b}, \frac{c}{d}$$

be two decimal fractions. Then

$$\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

may not be a decimal fraction.

Consider for example the decimal fractions

$$\frac{1}{4}, \frac{7}{10}.$$

We have
$$\frac{1}{4} \div \frac{7}{10} = \frac{1}{4} \times \frac{10}{7} = \frac{5}{14}.$$

Surely $\frac{5}{14}$ is not a decimal fraction in that, while it is in its lowest terms, the denominator admits of a prime factor 7 different from 2 and 5.

Note. It is not proposed to deal with the process of adding and multiplying decimal fractions given in decimal notation. In fact, it is only an extension of the process of adding and multiplying natural numbers given in the usual decimal notation.

EXERCISES

1. Arrange the following in ascending order

$$3{\cdot}273,\ 3{\cdot}365,\ 2{\cdot}476,\ 1{\cdot}587,\ 3{\cdot}373,\ 2{\cdot}374.$$

2. Replace the sign ? by equal to, greater than or less than to make the statement true.

(*i*) 2·732 ? 2·645 (*ii*) 1·317 ? 1·326

(*iii*) 9·123 ? 8·345 (*iv*) 7·234 ? 7·142.

26. ORDER-DENSENESS OF THE SET OF FRACTIONS

There is a property of the order relation in respect of the set of fractions which does not hold for the relation in respect of the set of natural numbers. This property is described as the **Order-denseness** of the set of fractions.

Before, however, stating this property, we introduce the notion of **Betweenness.**

Let

$$\frac{a}{b}, \frac{c}{d}$$

be two fractions such that

$$\frac{a}{b} < \frac{c}{d}\,;\ a, b, c\ d \in \mathrm{N}.$$

We say that a fraction

$$\frac{e}{f}$$

lies between the given fractions

$$\frac{a}{b}, \frac{c}{d}$$

if

$$\frac{a}{b} < \frac{e}{f} < \frac{c}{d}$$

Illustrations (*i*) $\frac{1}{4}$ lies between $\frac{1}{5}$ and $\frac{1}{2}$ and

(*ii*) 3·78 lies between 3·77 and 3 79.

We now state and prove a theorem.

Theorem. *Between any two different fractions there lies a fraction.*

Proof Let a/b, c/d be two given fractions. We shall show that the fraction

$$\frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right)$$

lies between the given fractions a/b, c/d.

We can assume without any loss of generality that

$$\frac{a}{b} < c$$

Now

$$\frac{a}{b} < \frac{c}{d}$$

$$\Rightarrow \frac{a}{b} + \frac{a}{b} < \frac{a}{b} + \frac{c}{d}$$

$$\Rightarrow \left(\frac{1}{1} + \frac{1}{1}\right)\frac{a}{b} < \frac{a}{b} + \frac{c}{d}$$

$$\Rightarrow \frac{2}{1} \cdot \frac{a}{b} < \frac{a}{b} + \frac{c}{d}$$

$$\Rightarrow \frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right).$$

Again

$$\frac{a}{b} < \frac{c}{d}$$

$$\Rightarrow \frac{a}{b} + \frac{c}{d} < \quad + \frac{c}{d}$$

$$\Rightarrow \frac{a}{b} + \frac{c}{d} < \left(\frac{1}{1} + \frac{1}{1}\right)\frac{c}{d}$$

$$\Rightarrow \frac{a}{b} + \frac{c}{d} < \frac{2}{1} \quad \frac{c}{d}.$$

$$\Rightarrow \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}.$$

Thus, we have shown that

$$\frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}.$$

Cor. *Between two different fractions there lies an infinite number of fractions.*

Let e/f be a fraction between the fraction a/b and c/d. There then also exists a fraction between a/b and e/f, say g/h.

We have

$$\frac{a}{b} < \frac{g}{h} < \frac{e}{} < \frac{c}{d}.$$

Obviously this process can be repeated an infinite number of times.

Hence the result.

This property is expressed by saying that the set of fractions is **Order-dense**.

Surely this property does *not* hold for the under relation in the set of natural numbers. For example, we have no natural number between the pairs,

$$3, 4 \cdot 5, 6$$

of natural numbers.

In fact, while we can talk of *consecutive natural numbers,* we cannot talk of *consecutive fractions.*

EXERCISES

1. Give any five fractions between

(*i*) $\frac{1}{3}$ $\frac{2}{3}$ (*ii*) $\frac{17}{1}$ $\frac{18}{1}$ (*iii*) $\frac{1}{5}$ $\frac{1}{4}$

(*iv*) $\frac{8}{9}$, $\frac{7}{8}$ (*v*) $\frac{13}{14}$, $\frac{11}{12}$ (*vi*) $\frac{17}{19}$, $\frac{9}{13}$

(*vii*) ·12, ·09 (*viii*) ·573, ·637.

2. Show that

$$\frac{1}{3}\left(\frac{2}{1} \quad \frac{a}{b} + \frac{c}{d}\right)$$

lies between $\frac{a}{b}$ $\frac{c}{d}$.

27. THE SET OF NATURAL NUMBERS AS A SUB-SET OF THE SET OF FRACTIONS

Usual Notation.

We associate with each natural number n the fraction $\frac{n}{1}$ so that whenever we come across the fraction $\frac{n}{1}$ or any fraction $\frac{kn}{k}$ equal to the same, we shall, if we so desire, replace it by n.

It has, however, to be seen that as a result we anticipate no confusion.

Consider the relations

$$k = m + n, \; l = m\,n$$

between the members

$$m, n, k, l$$

of N. It will be seen that the same equalities will persist if k, l, m, n are interpreted as

$$\frac{k}{1}, \frac{l}{1}, \frac{m}{1}, \frac{n}{1}$$

respectively. In fact, we have

$$\frac{m}{1} + \frac{n}{1} = \frac{m+n}{1} = \frac{k}{1}$$

$$\frac{m}{1} \cdot \frac{n}{1} = \frac{mn}{1} = \frac{l}{1}$$

Now suppose that

$$m > n.$$

Again, we have

$$\frac{m}{1} > \frac{n}{1} \Leftrightarrow m \,.\, 1 > n.\, 1 \Leftrightarrow m > n.$$

Thus, we agree to replace

$$\frac{m}{1} \text{ by } m$$

or *vice versa.*

Finally, we see that if $\frac{a}{b}$ be any fraction,

we have

$$\frac{a}{b} = \left(\frac{a}{1}\right) \div \left(\frac{b}{1}\right) = a \div b$$

so that each fraction a/b can be thought of as obtained on dividing a by b.

Thus, we can think of each natural number as a fraction in that we identify the natural number n with the fraction $n / 1$. From this point of view we see that the set N of natural numbers is a sub-set of the set F of fractions so that symbolically

$$\mathbf{N} \subset \mathbf{F}.$$

Of course, N is a proper sub-set of F *i.e.,* there are members of F which are not members of N as, for example, $2 / 3$, $7 / 9$.

28. SUMMARY

The set F of fractions is given by

$$\mathbf{F} = \left\{\frac{a}{b} : a \in \mathbf{N},\ b \in \mathbf{N}\right\}.$$

In the following, we shall denote a member of F by a single letter so that we shall denote a member of F by x, y, z, u, v, etc.

It should, of course, be kept in mind that each of x, y etc., is of the form a / b where a, b are natural numbers.

Addition Composition in **F.**

To each pair x, y of. members of F, corresponds a member of F denoted by $x + y$ and called the sum. Also the association of the member $x + y$ of F, to the pair x, y of members of F is called the addition composition in F.

The addition composition in F has the following properties.

1 Addition composition in F is commutative *i.e.,*

$$x + y = y + x \quad \forall\ x, y \in \mathbf{F}.$$

2. Addition composition in F is associative *i.e.,*

$$x + (y + z) = (x + y) + z \quad \forall\ x, y, z \in \mathbf{F}.$$

3 Addition composition in F satisfies the cancellation law *i.e.*,

$$x + z = y + z \Rightarrow x = y\ ,\ x, y, z \in \mathbf{F}.$$

Multiplication Composition in **F**.

To each pair x, y of members of F corresponds a member of F denoted by xy and called the product. Also the association of the member xy of F to the pair x, y of members of F is called the multiplication composition in F which has the following properties.

4. Multiplication composition in F is commutative *i.e.*,

$$xy = yx \ \forall\ x, y \in \mathbf{F}.$$

5. Multiplication composition in F is associative *i.e.*,

$$x\,(yz) = (xy)\,z \quad \forall\ x, y, z, \in \mathbf{F}.$$

6. The member, 1, of F is such that

$$x\,.\,1 = 1\,.\,x = x \quad \forall\ x \in \mathbf{F}.$$

The number, 1, is called the unity.

7. To each $x \in$ F corresponds $y \in$ F such that

$$xy = 1 = yx.$$

Each of x and y is called the *reciprocal* of the other.

8. Multiplication distributes addition, *i e.*,

$$x\,(y + z) = xy + xz \qquad \forall\ x, y, z \in \mathbf{F}.$$

Order Relation in **F**. There exists in F a relation 'Is greater than' having the following properties.

9 *Trichotomy Law.* If x, y be two members of F, we have one and only one of the following three possibilities :

(*i*) $x > y$ (*ii*) $y > x$ (*iii*) $x = y$.

10. *Transitivity Law*

$$x > y \text{ and } y > z \Leftrightarrow x > z\ ;\ x, y, z \in \mathbf{F}.$$

11. *Compatibility with Addition*

$$x > y \Leftrightarrow x + z > y + z\ ;\ x, y, z \in \mathbf{F}.$$

12. *Compatibility with Multiplication*

$$x > y \Leftrightarrow xz > yz\ ;\ x, y, z \in \mathbf{F}.$$

Division in **F**. If x, y be any two members of F, there exists $z \in$ F such that

$$x = yz.$$

We write

$$x \div y = z \ \text{ or } \ \frac{x}{y} = \text{ or } \ x/y = z$$

We have

$$x - y = z \Leftrightarrow x = yz.$$

Reciprocal of a Fraction. Let x be any fraction.

We have

$$x = \frac{a}{b},\ a, b \in \mathrm{N}.$$

The reciprocal of $\frac{a}{b}$ is the fraction $\frac{b}{a}$.

Now

$$\frac{1}{x} = \left(\frac{1}{1}\right) \div \left(\frac{a}{b}\right) = \frac{b}{a}$$

so that we see that the reciprocal of x can be described as

$$\frac{1}{x}.$$

Of course, we have

$$x \cdot \frac{1}{x} = 1.$$

We also show that

$$\frac{x}{y} = x \cdot \frac{1}{y}.$$

Now

$$\left(x \cdot \frac{1}{y}\right) y = x\left(\frac{1}{y} \cdot y\right) = x \cdot 1 = x$$

so that

$$\left(x \cdot \frac{1}{y}\right) y = x$$

$$\Rightarrow x \cdot \frac{1}{y} = \frac{x}{y}.$$

Subtraction in F. If x, y be two members of F such that $x > y$, then there exists $z \in \mathrm{F}$ such that

$$x = y + z,$$

We write

$$x - y = z.$$

We have

$$x = y + z \Leftrightarrow x - y = z.$$

Note. We have seen that when $a, b, c, d \in \mathrm{N}$

$$\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc.$$

It will now be shown that we have similar results when instead of natural numbers a, b etc., we have arbitrary numbers x, y etc., belonging to F.

Thus, we shall show that

$$\text{I} \qquad \frac{u}{x} = \frac{v}{y} \Leftrightarrow uy = vx$$

$$\text{II} \qquad \frac{u}{x} + \frac{v}{y} = \frac{uy + vx}{xy}$$

$$\text{III} \qquad \frac{u}{x} \cdot \frac{v}{y} = \frac{uv}{xy}$$

$$\text{IV} \qquad \frac{u}{x} > \frac{v}{y} \Leftrightarrow uy > vx$$

u, v, x, y denoting arbitrary members of F *i e.*, u, v, x, y are arbitrary fractions.

I. We have

$$\begin{aligned}
& \frac{u}{x} = \frac{v}{y} \\
\Leftrightarrow\ & u\frac{1}{x} = v\frac{1}{y} \\
\Leftrightarrow\ & u\frac{1}{x}(xy) = \left(v\frac{1}{y}\right)(xy) \\
\Leftrightarrow\ & u\left(\frac{1}{x}x\right)y = v\left(y\frac{1}{y}\right)x \\
\Leftrightarrow\ & u \,.\, 1 \,.\, y = v \,.\, 1 \,.\, x \\
\Leftrightarrow\ & uy = v\,x.
\end{aligned}$$

II. We have

$$\begin{aligned}
xy\left(\frac{u}{x} + \frac{v}{y}\right) &= xy\left(\frac{u}{x}\right) + xy\left(\frac{v}{y}\right) \\
&= (xy)\left(u\frac{1}{x}\right) + (xy)\left(v\frac{1}{y}\right) \\
&= (yx)\left(\frac{1}{x} \,.\, u\right) + (xy)\left(\frac{1}{y}\,v\right) \\
&= y\left(x \,.\, \frac{1}{x}\right)u + x\left(y \,.\, \frac{1}{y}\right)v. \\
&= y \,.\, 1 \,.\, u + x \,.\, 1 \,.\, v. \\
&= uy + xv
\end{aligned}$$

implying, by the definition of division,

$$\frac{u}{x} + \frac{v}{y} = \frac{yu + xv}{xy} = \frac{uy + vx}{xy}.$$

III. We have

$$\frac{u}{x} \cdot \frac{v}{y} = \left(u \cdot \frac{1}{x}\right)\left(v \cdot \frac{1}{y}\right)$$
$$= u\left(\frac{1}{x} \cdot v\right)\frac{1}{y}$$
$$= u\left(v \cdot \frac{1}{x}\right)\frac{1}{y}$$
$$= uv\left(\frac{1}{x} \cdot \frac{1}{y}\right).$$

Also we have

$$(xy)\left(\frac{1}{x} \cdot \frac{1}{y}\right) = xy\left(\frac{1}{y} \cdot \frac{1}{x}\right) = x\left(y \cdot \frac{1}{y}\right)\frac{1}{x}$$
$$= x \cdot 1 \cdot \frac{1}{x} = x \cdot \frac{1}{x} = 1$$

implying

$$\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy}.$$

Thus, we have

$$\frac{u}{x} \cdot \frac{v}{y} = uv \cdot \frac{1}{xy} = \frac{uv}{xy}.$$

IV. We have

$$\frac{u}{x} > \frac{v}{y}$$
$$\Leftrightarrow xy\left(\frac{u}{x}\right) > (xy)\frac{v}{y}$$
$$\Leftrightarrow (xy)\left(u \cdot \frac{1}{x}\right) > (xy)\left(v \cdot \frac{1}{y}\right)$$
$$\Leftrightarrow (yx)\left(\frac{1}{x} \cdot u\right) > (xy)\left(\frac{1}{y} \cdot\right)v$$
$$\Leftrightarrow y\left(x \cdot \frac{1}{x}\right)u > x\left(y \cdot \frac{1}{y}\right)v$$
$$\Leftrightarrow y \cdot 1 \cdot u > x \cdot 1 \cdot v$$
$$\Leftrightarrow yu > xv$$
$$\Leftrightarrow uy > vx.$$

29. ALGEBRAIC EXPRESSIONS

An expression consisting of numbers and letters as variables with some number domains combined with each other by means of addition, subtraction,

multiplication and division operations is called an *algebraic expression*. Here we shall consider expressions involving variables with the set F of fractions as the domain.

We may repeat that symbol

$$x^n$$

where $n \in \mathbf{N}$ and $x \in \mathbf{F}$

denote the product of x with itself taken n times.

Thus,

$$\underbrace{x^n = x.\,x\,.\,x.\ldots x}_{n\text{-times}},\ x \in \mathbf{F}, n \in \mathbf{N}.$$

We give below a few algebraic expressions.

(*i*) $2x^2 + \frac{3}{4}x + \frac{7}{11}$

(*ii*) $\frac{x+y}{x^2+y^2}$

(*iii*) $\frac{\cdot 32x + \frac{2y}{3}}{5x^2 + 7y^3}$.

It may be remarked that, often, it is possible, with the help of the rules of addition and multiplication, to express an algebraic expression in a simpler form.

Examples

1. Write as an indicated sum.

$$(x^2 + 11x + 24)(x + 4) \text{ where } x \in \mathbf{F}.$$

In doing so, we shall in the following make use of the CAD laws in F.

We have

$$\begin{aligned} x^2 + (11x + 24)(x + 4) &= (x^2 + 11x + 24)\,x + (x^3 + 11x + 24)\,4 \\ &= (x^2.\,x + 11\,x\,.\,x + 24\,.\,x) + (x^2.\,4 + 11x\,.\,4 + 24\,.\,4) \\ &= (x^3 + 11x^2 + 24x) + (4x^2 + 44x + 96) \\ &= x^3 + (11 + 4)\,x^2 + (24 + 44)\,x + 96 \\ &= x^3 + 15\,x^2 + 68x + 96. \end{aligned}$$

2. Show that

$$\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x^2} + \frac{1}{y^2}} = \frac{xy\,(x+y)}{x^2+y^2} \ \forall\ x, y \in \mathbf{F}.$$

We have

$$\frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}+\dfrac{1}{y^2}}=\frac{\dfrac{y+x}{xy}}{\dfrac{y^2+x^2}{x^2y^2}}$$

$$=\frac{y+x}{xy}\cdot\frac{x^2y^2}{y^2+x^2}$$

$$=\frac{(x+y)\,x^2y^2}{(x^2+y^2)\,xy}$$

$$=\frac{(x+y)\,xy}{x^2+y^2}=\frac{xy\,(x+y)}{x^2+y^2}.$$

3. Simplify

$$\left(x+\frac{1}{y}\right)\div\left(y+\frac{1}{x}\right)\quad\text{where } x, y\in \mathbf{F}.$$

We have

$$\left(x+\frac{1}{y}\right)\div\left(y+\frac{1}{x}\right)=\left(\frac{x}{1}+\frac{1}{y}\right)\div\left(\frac{y}{1}+\frac{1}{x}\right)$$

$$=\frac{xy+1}{y}\div\frac{yx+1}{x}$$

$$=\frac{xy+1}{y}\times\frac{x}{yx+1}$$

$$=\frac{x\,(xy+1)}{y\,(xy+1)}=\frac{x}{y}.$$

4. Show that

$$(x+3)+\frac{4x+3}{x^2+1}=\frac{x^3+3x^2+5x+6}{x^2+1}\ \forall\ x\in\mathbf{F}.$$

We have

$$(x+3)+\frac{4x+3}{x^2+1}=\frac{x+3}{1}+\frac{4x+3}{x^2+1}$$

$$=\frac{(x+3)\,(x^2+1)+4x+3}{x^2+1}$$

$$=\frac{[x\,(x^2+1)+3\,(x^2+1)\,]+4x+3}{x^2+1}$$

$$=\frac{[\,(x^3+x)+(3x^2+3)\,]+4x+3}{x^2+1}$$

$$=\frac{x^3+3x^2+5x+6}{x^2+1}.$$

5. Simplify

$$\left(x+\frac{1}{2}\right)\left(\frac{3}{4x}+\frac{5}{x^2}\right)\quad\text{where } x\in\mathbf{F}.$$

We have

$$\left(x+\frac{1}{2}\right)\left(\frac{3}{4x}+\frac{5}{x^2}\right)=\left(\frac{x}{1}+\frac{1}{2}\right)\left(\frac{3x+20}{4x^2}\right)$$

$$=\frac{2x+1}{2}\cdot\frac{3x+20}{4x^2}$$

$$=\frac{(2x+1)(4x+20)}{8x^2}$$

$$=\frac{(2x+1)\,3x+(2x+1)\,20}{8x^2}$$

$$=\frac{(6x^2+3x)+(40x+20)}{8x^2}$$

$$=\frac{6x^2+(3+40)\,x+20}{8x^2}$$

$$=\frac{6x^2+43x+20}{8x^2}.$$

EXERCISES

1. x, y, z......being members of F, write the following as indicated sums.

(*i*) $\left(\frac{1}{2}x+3\right)\left(x+5\right)$ (*ii*) $\left(\frac{3}{4}x+\frac{5}{7}y\right)\left(\frac{2}{3}x+\frac{7}{10}y\right)$

(*iii*) $({\cdot}01x+{\cdot}37z)\,({\cdot}5x+{\cdot}15y)$ (*iv*) $\left(z+{\cdot}5\right)\left(\frac{1}{3}z+\frac{1}{4}\right)$

(*v*) $\left({\cdot}5x+3y\right)\left(1{\cdot}3y+\frac{2}{3}z\right)$ (*vi*) $\left(x+\frac{7}{3}\right)\left(\frac{1}{2}x^2+\frac{5}{13}\right)$

(*vii*) $\frac{1}{xyz}\left(x^2yz+{\cdot}5xy^2z+\frac{7}{2}xyz^2\right)$

(*viii*) $\left({\cdot}3x+\frac{1}{7}y\right)\left(x^2+1{\cdot}5y^2\right)$

(*ix*) $xyz^2\left(\frac{2}{3}x+\frac{5}{4}y+\frac{7}{3}\cdot\frac{1}{z}\right)$

(*x*) $x^2y^2z^2\left(\frac{2}{9x}+\frac{14}{5y}+\frac{12}{3z}\right)$.

2. Show that the following statements are true, whatever fractions x, y, z may be.

(*i*) $\frac{x}{3}+\frac{y}{7}=\frac{7x+3y}{21}$ (*ii*) $\frac{5}{xy+y^2}+\frac{2}{x^2+xy}=\frac{5x+2y}{xy\,(x+y)}$

(*iii*) $\frac{x^2y}{xy+y^2+yz}+\frac{y^2x}{x^2+xy+xz}+\frac{z^2}{z^2+zx+zy}=\frac{x^2+y^2+z^2}{x+y+z}$

(iv) $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^3}+\dfrac{1}{y^3}} = \dfrac{x^2y^2(x+y)}{x^3+y^3}$ (v) $\dfrac{\dfrac{1}{x^2}+\dfrac{1}{y^2}}{\dfrac{1}{x^3}+\dfrac{1}{y^3}} = \dfrac{xy(x^2+y^2)}{x^3+y^3}$

(vi) $\dfrac{1}{x+5}+\dfrac{2}{3(x+5)^2}=\dfrac{3x+17}{3(x+5)^2}$

(vii) $\dfrac{a}{x+b}+\dfrac{c}{x+d}=\dfrac{(a+c)\,x+ad+bc}{(x+b)\,(x+d)}$

(viii) $\dfrac{3}{x+1}+\dfrac{2x+1}{x^2+1}=\dfrac{5x^2+3x+4}{(x+1)(x^2+1)}$

(ix) $\dfrac{2x+1}{x^2+1}+\dfrac{3x+5}{x^2+2}=\dfrac{5x^3+6x^2+7x+7}{(x^2+1)\,(x^2+2)}$

(x) $\dfrac{2}{3x+4}+\dfrac{3}{2x+5}=\dfrac{13x+22}{(3x+4)(2x+5)}$.

3. Simplify the following, x, y, z......being members of F.

(i) $1 \div \left(\dfrac{1}{x}+\dfrac{1}{y}\right)$ (ii) $\left(2x+\dfrac{3}{y^2}\right) \div \left(3x+\dfrac{2}{y^3}\right)$

(iii) $\left(\dfrac{2}{x}+\dfrac{3}{y}\right) \div \left(\dfrac{5}{x}+\dfrac{2}{y}\right)$ (iv) $\left(x+\dfrac{\cdot 5}{y}\right) \div \dfrac{10}{xy}$

(v) $\left(1+\dfrac{1}{x}\right) \div \left(1+\dfrac{1}{x^2}\right)$ (vi) $\dfrac{6z+12}{5} \times \dfrac{15y}{7z+14}$

(vii) $\dfrac{4x^2+6}{8x+6}$ (viii) $\dfrac{2y}{3}+\dfrac{1}{3y}$

(ix) $\dfrac{y}{2}+\dfrac{1}{2y}$ (x) $\dfrac{2y}{7z^2} \times \dfrac{3yz}{8} \times \dfrac{yz}{9a^2}$

(xi) $\dfrac{x}{x^2+2x}+\dfrac{3}{x}$ (xii) $\dfrac{xyz+\dfrac{1}{2}x^2y}{\dfrac{3}{4}x+\dfrac{3}{2}z}$

(xiii) $\dfrac{\dfrac{7}{11}x+\dfrac{13}{8}y}{\dfrac{8}{13}x+\dfrac{11}{7}y}$ (xiv) $\dfrac{x^3+y^3}{\dfrac{1}{x^3}+\dfrac{1}{y^3}}$

(xv) $\dfrac{3x^2+5y^2}{\cdot 5x+\dfrac{3}{4}y} \times \dfrac{\dfrac{\cdot 5}{y}+\dfrac{3}{4}\cdot\dfrac{1}{x}}{\dfrac{1}{5}x^2+\dfrac{1}{3}y^2}$

(xvi) $\dfrac{\frac{3}{7}x^2 + \frac{2}{3}y^2}{\cdot 75x + \frac{3}{11}y} \times \dfrac{\frac{8\,25}{y} + \frac{3}{x}}{\frac{7}{13}x^3 + \frac{19}{12}y^3} \times \dfrac{\frac{12}{13}x^3 + \frac{19}{7}y^3}{9x^2 + 14y^2}$

4. Show that $3x + 4 > 5x + 2 \; \forall \; x < 1 \; ; x \in \mathbf{F}$.

5. Show that $x_2 > x_1 \Rightarrow x_2^2 > x_1^2 \; ; x_1, x_2 \in \mathbf{F}$.

6. Show that $x_2 < x_1 \Rightarrow \dfrac{1}{x_2} > \dfrac{1}{x_1} \; ; x_1 , x_2 \in \mathbf{F}$.

7. Show that $\dfrac{1}{x+3} - \dfrac{1}{x+4} = \dfrac{1}{(x+3)(x+4)} \; \forall \; x \in \mathbf{F}$.

8. Show that $\dfrac{1}{2x^2+5} > \dfrac{1}{2x^2+7} \; \forall \; x \in \mathbf{F}$.

Also show that $\dfrac{1}{2x^2+5} - \dfrac{1}{2x^2+7} = \dfrac{2}{(2x^2+5)(2x^2+7)} \; \forall \; x \in \mathbf{F}$.

30. OPEN STATEMENTS

Examples

1. Find the truth set of

$$3x + 5 = 7 \; ; x \in \mathbf{F}.$$

Solution. Our attempt is to see that the term containing x occurs on one side of the equality and the term involving a number on the other.

We have $\quad 3x + 5 = 7$

or equivalently $\quad 3x + 5 = 2 + 5.$

Cancelling 5 from both sides, we obtain

$$3x = 2.$$

Multiplying both sides by $\frac{1}{3}$, we obtain

$$\frac{1}{3} \cdot (3x) = \frac{1}{3} \cdot 2$$

or equivalently

$$x = \frac{2}{3}.$$

The procedure may be exhibited as follows :

$$\begin{aligned} & 3x + 5 = 7 \\ \Leftrightarrow \; & 3x + 5 = 2 + 5 \\ \Leftrightarrow \; & 3x = 2 \\ \Leftrightarrow \; & \frac{1}{3}(3x) = \frac{1}{3} \cdot 2 \\ \Leftrightarrow \; & x = \frac{2}{3}. \end{aligned}$$

The required truth set is

$$\left\{\frac{2}{3}\right\}$$

consisting of only one member.

Note. It will be seen that we have tried to obtain a chain of statements each equivalent to the other such that the truth set of the last member of the chain of equivalent statements can be put down at sight.

2. *Find the truth set of*

$$5x + 4 = 3x + 2\ ;\ x \in \mathbf{F}.$$

Solution. We obtain a chain of equivalent statements. We have

$$5x + 4 = 3x + 2$$
$$\Leftrightarrow 3x + (2x + 4) = 3x + 2$$
$$\Leftrightarrow 2x + 4 = 2.$$

Now we have

$$4 > 2$$
$$\Rightarrow 2x + 4 > 2\ \forall\ x \in \mathbf{F},$$

so that there exists no member x of $\mathbf{F}$ such that

$$2x + 4 = 2.$$

Thus, the truth set of the given open statement, in respect of the set of fractions, is void.

3. *Find the truth set of the inequality*

$$3x + 2 < 5,\ x \in \mathbf{F}.$$

Solution. We have

$$3x + 2 < 5$$
$$\Leftrightarrow 3x + 2 < 3 + 2$$
$$\Leftrightarrow \quad 3x < 3$$
$$\Leftrightarrow \quad x < 1.$$

In particular, we see that the fractions

$$\frac{1}{2},\ \frac{1}{3},\ \frac{4}{5}$$

satisfy the given open statement.

1. Solve for x, given that $x \in \mathbf{F}$.

(*i*) $x + \frac{11}{7} = \frac{13}{7}$ (*ii*) $x + \frac{2}{3} = \frac{8}{5}$

(*iii*) $2x = 3$ (*iv*) $\frac{5}{7}\ x = \frac{19}{17}$

(v) $4 = 9x$

(vi) $x - \frac{6}{11} = \frac{5}{2}$

(vii) $\frac{15}{4} - x = \frac{1}{3}$

(viii) $2x + 3 = 4$

(ix) $6 + 7x = 9$

(x) $2x + 7 = 3x + 2$

(xi) $17x - 2 = 1$

(xii) $7 - 5x = 3$

(xiii) $5x + 2 = 5 + 3x$

(xiv) $12 - 3x = x + 3$

(xv) $7x - 2 = 2 - 3x$.

2. Given $x \in \mathbf{F}$, find the truth sets of the following equations.

(i) $13x + 4 = 5x + 12$

(ii) $5 - 11x = x + 5$

(iii) $25 - x = 4 + 11x$

(iv) $4 - 3x = x + 11$

(v) $4x + 3 = 1$

(vi) $10x + 3 = 23$

(vii) $19 - 5x = 2x + 5$

(viii) $23 - 2x = 3x + 25$.

(ix) $3x + 4 = 4 - 7x$

(x) $25 + 7x = 2x + 15$

(xi) $4x + 6 = 2x + 25$.

3. Given $x \in \mathbf{F}$, find the truth sets of the following open statements.

(i) $x \div 3 = \frac{2}{7}$

(ii) $x \div \frac{3}{8} = \frac{13}{11}$

(iii) $3 \div x = 6$

(iv) $25 \div (2x) = 4$

(v) $(x \div 5) + 2 = 7$

(vi) $16 \div x = 4$

(vii) $(x \div 3) - 2 = 2x - 7$

(viii) $2 - (3x \div 4) = x + \frac{5}{2}$.

4. Find the truth sets of the following open statements, given $x \in \mathbf{F}$.

(i) $3x - 2 < 4$

(ii) $2 - 5x > \frac{1}{2}$

(iii) $2x \div 3 > 5$

(iv) $2x \div 3 < 4$

(v) $\frac{3}{4}x + \frac{1}{2} > \frac{9}{16}$

(vi) $13x + \frac{7}{2} < \frac{47}{15}$

(vii) $2x + 3 \leqslant 6$

(viii) $\frac{5}{3}x + \frac{2}{7} < \frac{1}{10}$

(ix) $\frac{12}{13}x + \frac{1}{2} < \frac{3}{4}$

(x) $4x + 2 \geqslant 3 + 2x$

(xi) $7 + 3x \geqslant 5x + 4$

(xii) $7 - 3x \geqslant 5x + 4$

(xiii) $x + 11 \leqslant 3x + 10$

(xiv) $4x + 15 < 3x + 5$

(xv) $3 \div x \leqslant 5$.

PROBLEMS

1. Find the number, such that two-thirds of its square equals 7 times the number.

Solution.

Let the number be x.

Then two-thirds of its square $= \frac{2}{3} x^2$.

Also seven times the number is $7x$.

$$\therefore \quad \frac{2}{3} x^2 = 7x$$
$$\Leftrightarrow \quad \frac{2}{3} x = 7$$
$$\Leftrightarrow \quad \frac{3}{2} \times \left(\frac{2}{3} \times x\right) = \frac{3}{2} \times 7$$
$$\Leftrightarrow \quad x = \frac{21}{2}.$$

The required number, therefore, is $\frac{21}{2}$.

2. Ram can finish a piece of work in 6 days and Krishan can complete the same work in 12 days. In how many days will they be able to finish the work, if both do the job together ?

Solution. Let x be the number of days in which both Ram and Krishan together can finish the work.

Now in one day Ram is able to finish $\frac{1}{6}$ of the work.

In x days Ram is able to finish $\frac{1}{6} \times x$ of the work.

Again, in one day Krishan is able to finish $\frac{1}{12}$ of the work. In x days Krishan is able to finish $\frac{1}{12} \times x$ of the work.

As, by the assumption, they both together finish the work in x days, we have

$$\frac{1}{6} x + \frac{1}{12} x = 1$$
$$\Leftrightarrow 12\left(\frac{1}{6} x + \frac{1}{12} x\right) = 12$$
$$\Leftrightarrow \quad 2x + x = 12$$
$$\Leftrightarrow \quad 3x = 12$$
$$\Leftrightarrow \quad x = 4.$$

They, together, are able to finish the work in 4 days.

3. A man invests a total of Rs. 4,000, partly at $6\frac{1}{4}\%$ and the remaining at $6\frac{1}{2}\%$. If his total income is Rs. 255·675, find his two investments.

Solution. Let the first investment be x rupees. Then, the income from this investment $= \frac{x}{100} \times 6\frac{1}{4}$ rupees.

Also, the second investment will be Rs. $(4000 - x)$.

Income from the second investment $= \frac{4000 - x}{100} \times 6\frac{1}{2}$ rupees. But it is given that the total income is Rs. 255·675.

$$\therefore \quad \frac{x}{100} \times 6\frac{1}{4} + \frac{4000 - x}{100} \times 6\frac{1}{2} = 255{\cdot}675$$

$$\Leftrightarrow \quad \frac{25x}{400} + \frac{13(4000 - x)}{200} = 255{\cdot}675$$

$$\Leftrightarrow \quad 400\left[\frac{25x}{400} + \frac{13(4000 - x)}{200}\right] = 400 \times (255{\cdot}675)$$

$$\Leftrightarrow \quad 25x + 104000 - 26x = 102270$$

$$\Leftrightarrow \quad 25x + 104000 - 26x + 26x = 102270 + 26x$$

$$\Leftrightarrow \quad 1730 = x.$$

His first investment is Rs. 1730 and the second investment is Rs. (4000 — 1730), *i.e.*, Rs. 2270.

EXERCISES

1. What number divided by a number 10 less than itself gives the result 6 ?

2. One number is 10 less than three times a smaller number. Dividing the larger number by 8 gives the same result as dividing the smaller by 3. What are the numbers ?

3. Two-thirds the sum of a certain number and 21 equals 30 Find the number.

4. One number is 5 more than twice another. The ratio of the numbers is 15 : 7. Find the numbers.

5. The tail of an aeroplane measures $\frac{1}{7}$ of its total length and the cockpit measures $\frac{1}{8}$ of its total length. What fraction of the total length is the rest of the body of the plane ?

6. The total income of a family per month is Rs. 500. One-forth of it is given as rent, one-tenth is spent on clothing and three-eighths on food. How much is left for other purposes

7. Ram sleeps for $\frac{1}{3}$, eats for $\frac{1}{9}$ and studies for $\frac{1}{4}$ of the day. How many hours have not been accounted for ?

8. Krishan planned a three-day trip in a car, intending to cover $\frac{1}{3}$ of the total distance each day. Because of some trouble in the engine of the car, he could cover only $\frac{1}{5}$ of the total distance on the second day. What fraction of the journey must be covered on the the last day to arrive in time.

9. Two astronauts, Pat and Mike, were orbiting the earth in separate space capsules. They were orbiting in the same direction and in the same plane. Pat orbits in 3 hours and Mike in $7\frac{1}{2}$ hours. At 12 noon, Delhi time, Mike sees Pat directly below. At what time will Pat and Mike be one above the other again ?

10. One painter works $\frac{5}{3}$ times as fast as another. How long will it take the slower one to paint a house which takes them 6 days together ?

11. Three pipes can fill a tank in 3, 4 and 5 hours separately. How long will it take them to fill the tank together ?

12. A mason can build a wall in 12 days. But with another assistant given, the job is finished in 8 days. How long will it take the assistant to build the wall single handed ?

13. Ten litres of alcohol, used in the laboratory is 80% pure (*i.e.*, 80% alcohol, 20% water). How many litres of water must be added so that the resulting mixture contains 30% alcohol ?

14. Mohan invests Rs. 10,000. On a part, he incurs a loss of 4% and on the remaining, he gains 5%. Find his two investments, if on the whole he neither gains nor loses.

15. Distribute Rs. 790 amongst X, Y and Z so that Y gets 20% more than X and 25% more than Z.

REVIEW EXERCISES

1. Given the sets

$$A = \left\{\frac{3}{5}, \frac{7}{9}, 2, \frac{23}{5}, 3, \frac{11}{2}, \frac{22}{26}\right\},$$

$$B = \left\{\frac{11}{13}, 3, \frac{17}{4}, 5, \frac{21}{8}, \frac{57}{15}, \frac{6}{10}\right\},$$

find the greatest and least members of the sets,

$$A, B, A \cup B \text{ and } A \cap B.$$

2. Given

$L = \{x : 1 < x < 2 \text{ and } x \in \mathbf{F}\}$,
$M = \{x : 1 \leqslant x < 2 \text{ and } x \in \mathbf{F}\}$,
$N = \{x : 1 < x \leqslant 2 \text{ and } x \in \mathbf{F}\}$,
$P = \{x : 1 \leqslant x \leqslant 2 \text{ and } x \in \mathbf{F}\}$,

obtain the greatest and least members, if they exist, of the sets

$$L, M, N, P$$
$$L \cup M, L \cup N, M \cup N.$$

3. Show that
$$x > y \Rightarrow x^3 > y^3 \qquad x, y \in \mathbf{F}.$$

4. Prove that
$$x^2 + y^2 > xy \qquad x, y \in \mathbf{F}.$$

5. Show that
$$x > y \Rightarrow \frac{x}{y} > \frac{x+1}{y+1} \qquad x, y \in \mathbf{F}.$$

6. Prove that
$$x > y \Rightarrow x^3 + 3x^2y > 3x^2y + y^3 \qquad x, y \in \mathbf{F}.$$

7. Given that
$$x > y\,;\; x, y \in \mathbf{F}.$$

Prove that,

(i) $\dfrac{1}{y^2} > \dfrac{1}{x^2}$

(ii) $\dfrac{1}{2y+5} > \dfrac{1}{2x+5}$

(iii) $\dfrac{1}{7y^3} > \dfrac{1}{7x^3}$.

8. Write as indicated sum, x, y, z, a, b, c being members of $\mathbf{F}$.

(i) $(ax^2 + by^2)(ay^2 + bz^2)$ (ii) $(ax + by + cz)(x + y + z)$

(iii) $\left(\cdot 5a + \dfrac{1}{3}b\right)\left(x + 2y + 3z\right)$

(iv) $\left(1\cdot 7x + 2\cdot 3y\right)\left(a + \dfrac{2}{5}z\right)$

(v) $xy^2z^2\left(\dfrac{3}{z} + \dfrac{2}{y^2} + \dfrac{1}{7}x\right)$.

9. Simplify the following, x, y, z, a, b, c being members of $\mathbf{F}$.

(i) $\dfrac{\dfrac{7}{3}\dfrac{x}{y} + 5}{x + \dfrac{15}{7}y}$ (ii) $\dfrac{x + \dfrac{1}{x}}{1 + \dfrac{1}{x^2}}$

(iii) $\left(\dfrac{5}{4}x + \dfrac{2}{3}y\right) \div \left(\dfrac{3}{4}x + \dfrac{2}{5}y\right)$

(iv) $\dfrac{3a + 4b + \dfrac{1}{2} c}{a^2 + \dfrac{4}{3} ab + \dfrac{1}{6} ac}$ (v) $\dfrac{1 + \dfrac{z}{xy}}{\dfrac{xy}{z} + 1}$

(vi) $\dfrac{ax + by}{cz^2 + az} \times \dfrac{a + cz}{\dfrac{x}{b} + \dfrac{y}{a}}$ (vii) $\dfrac{\cdot 4x + \cdot 15y}{8x + 3y}$

(viii) $\dfrac{x + \dfrac{1}{x}}{y + \dfrac{1}{y}} \div \dfrac{1 + \dfrac{1}{x^2}}{\dfrac{1}{y^2} + 1}$ (ix) $\dfrac{x^2 + \dfrac{1}{x}}{z + \dfrac{1}{z^2}} \times \dfrac{z^2 + \dfrac{1}{z}}{x + \dfrac{1}{x^2}}$

(x) $\dfrac{ax + bx}{c(y + z)} \times \dfrac{a(z + x)}{bz + az} \times \dfrac{\dfrac{2}{3}\left(y + z\right)}{\dfrac{2}{5} z + \cdot 4x}$.

10. Given $x \in F$, find the truth sets of the following open statements.

(i) $27 - 3x = 2x + 21$ (ii) $22 \div (x + 3) = 5$

(iii) $(7 \div x) + 5 = \dfrac{21}{4}$ (iv) $\dfrac{7}{5} + \dfrac{2}{3} x = \dfrac{15}{16} x + \dfrac{11}{13}$

(v) $[(x - 3) \div 2] + 3 = \dfrac{2x + 7}{4}$

(vi) $\dfrac{2}{3} x + \dfrac{11}{4} < \dfrac{21}{5}$ (vii) $\dfrac{3}{x} + 5 \geqslant 6{\cdot}5$

(viii) $\dfrac{x}{2} + \dfrac{3}{8} \leqslant \dfrac{27}{72}$ (ix) $(x \div 5) - 3 \leqslant \dfrac{1}{2}$

(x) $(27 \div x) + 3 \geqslant \dfrac{11}{7}$.

11. A chemist has a solution of 50% pure acid and another of 80% pure acid. How many grammes of each will make 600 grammes of a solution with 72% purity of the acid ?

12. A solution contains 40 grammes of sugar and 200 grammes of water. How much sugar must be added to make a 50% sugar solution ?

13. In 12 minutes an air-conditioner lowers the temperature by 10 degrees. However, if another air-conditioner is switched on, the two together take 4 minutes to lower the temperature by 10 degrees. How long will the second alone take to produce the same change ?

14. A job can be done by 8 men in 3 hours, or by 15 boys in 5 hours. How long will it take 3 men and 25 boys together ?

15. A man can swim 4 kilometres an hour in still water. In going 4 kilometres upstream, he takes the same as he takes in swimming 12 kilometres downstream. What is the speed of the water in the stream ?

16. A man invested some money at 5% and Rs. 800 less at $3\frac{1}{2}$%. In all, he received Rs. 210 from the two investments. Find his two investments.

17. A man invests a certain sum at 4% per annum and another sum which is Rs. 500 more than the first at 5% per annum. The interest on the second exceeds that on the first by Rs. 33 a year. Find his investments.

18. A man leaves 60% of his property to his wife and the rest to his son. The wife invests at 5 5% per annum and the son at 4·5% per annum. Find the yearly income of the wife, if that of the son is Rs. 5,400.

19. Even after allowing 10% discount on the marked price of an article, a shopkeeper gains 10%. Find the cost price of the article if its marked price is Rs. 77.

20. A man purchases two horses for a total of Rs. 1,000. He sells one at a profit of 30% and the second at a loss of 20%. In the whole transaction he gains Rs. 100. Find his cost of the two horses.

4

Rational Numbers

31. INTRODUCTION

It has been seen that the set F of fractions is *closed* not only for the addition and multiplication compositions but also for the division composition as inverse of the multiplication composition. This means that

$$x \in \mathbf{F},\ y \in \mathbf{F} \Rightarrow \begin{cases} x + y \in \mathbf{F} \\ x \times y \in \mathbf{F} \\ x \div y \in \mathbf{F}. \end{cases}$$

The trouble, however, still remains *vis - a - vis* the inverse of the addition composition in that x, y being members of **F**, the symbol

$$x - y$$

is meaningful in respect of the set of fractions *if and only if*

$$x > y.$$

For example, while the expression

$$\frac{7}{8} - \frac{1}{5},$$

is meaningful, the expression

$$\frac{1}{5} - \frac{7}{8},$$

is not, in respect of the set of fractions inasmuch as

$$\frac{7}{8} > \frac{1}{5}.$$

It is now proposed to *invent* new numbers and to have a richer set of numbers, to be known as the set of *rational numbers* to remove the restriction on subtraction as well. This new set which is a super-set of the set of fractions is such

that subtraction in respect of the same is meaningful for any pair of members of the same. Thus the purpose of this chapter is

(1) to set up the set of rational numbers,

(2) to define the addition and multiplication compositions, and the 'Is greater than' relation in the set of rational numbers,

(3) to develop the properties of the two compositions and the relation referred to in (2) above, and

(4) to consider the compositions of subtraction and division.

It will be found useful for this purpose to introduce the notion of what may be called

Signed Numbers.

32. THE NOTION OF SIGNED NUMBERS

In our everyday life, we have occasion to talk of pairs of entities such that of the two members of a pair one can be thought of as some sort of an opposite of the other Thus, for example, we talk of

(*i*) Income and expenditure

(*ii*) Profit and loss

(*iii*) Rise and fall

(*iv*) Movement towards east and movement towards west.

Let us consider some one having a profit of Rs. 200 or a loss of Rs. 200.

If we agree to denote the profit of Rs. 200 as profit of

$$+ 200 \text{ rupees}$$

then the loss of Rs. 200 may be described as a profit of

$$- 200 \text{ rupees.}$$

Again, if we denote the velocity of a train moving towards the east at the rate of 40 kilometres an hour as a velocity of

$$+ 40 \text{ kilometres per hour,}$$

we shall denote the velocity of a train moving towards west at 40 kilometres an hour as a velocity of

$$- 40 \text{ kilometres per hour.}$$

It may be remarked that nothing prevents us from changing the nomenclature and denoting the loss of Rs. 200 as a loss of $+ 200$ rupees, and the profit of Rs. 200 as a loss of -200 rupees.

It is important to note in this context that we are essentially concerned with *three* symbols *viz.*,

$$200, + 200, - 200$$

corresponding respectively to

(*i*) the amount of 200 rupees,

(*ii*) the profit of + 200 rupees,

(*iii*) the profit of − 200 rupees.

We shall refer to

$$+ 200, \ - 200$$

as two *signed numbers* and describe them as *positive* and *negative* numbers respectively.

The number 200 will be referred to as the *absolute value* of each of the two signed numbers

$$+ 200, \ - 200.$$

It may be remembered that the signs +, − prefixed to 200 are to be thought of as integral parts of the signed numbers and that these symbols are here being used to serve a purpose different from the one for which they were used earlier. We had earlier used these signs for denoting the addition and the subtraction operations and in this context they were placed *between* two numbers as

$$7 + 5, \ 7 - 5$$

and not just prefixed to individual numbers as we have now done in respect o + 200 and − 200.

Now, in the next section, we define the set of rational numbers.

33. THE SET OF RATIONAL NUMBERS

To each

$$\frac{a}{b} \in \mathbf{F}$$

we associate the two signed numbers

$$+ \frac{a}{b}, \ - \frac{a}{b}$$

and refer to the same as **Rational Numbers.**

Besides these, we also introduce the symbol

'0'

to be called the number **zero.**

We shall refer to

$$+ \frac{a}{b}$$

as a *positive rational number* and

$$- \frac{a}{b}$$

as a *negative rational number*.

The number, 0, will be considered as neither positive nor negative. Thus, a rational number may be positive, negative or zero.

Moreover, we shall *identify* each of

$$+0, -0$$

with

$$0.$$

Thus, the following are some of the rational numbers :

$$+3, -7, -\frac{3}{8}, +\frac{11}{12}, +\frac{23}{12}, -\frac{11}{6}, -\frac{2}{3}.$$

Of these

$$+3, +\frac{11}{12}, +\frac{23}{12}$$

are positive rational numbers and

$$-7, -\frac{3}{8}, -\frac{11}{6}, -\frac{2}{3}$$

are negative rational numbers.

The set of rational numbers will be denoted by the symbol

$$\mathbf{Q}$$

which is the first letter of the word '*Quotient*'. This letter, **Q**, is suggested by the fact that each rational number, other than zero, arises as a *quotient* of two natural numbers, prefixed with the sign ' + ' or the sign ' − '.

It may perhaps be thought that the set of rational numbers should be denoted by **R**. It needs, therefore, to be remarked that the letter **R** has been reserved for the set of real numbers which is a further development of the set of rational numbers.

Thus, we have

$$\mathbf{Q} = \{ +x, -x, 0 : x \in \mathbf{F} \}.$$

There is an important sub-set of **Q** called the set of **Integers** which we shall denote by **I**. We have

$$\mathbf{I} = \{ +x, -x, 0 ; x \in \mathbf{N} \}.$$

We may also describe **I** as follows :

$$\mathbf{I} = \{ 0, +1, -1, +2, -2, +3, -3, . \}.$$

Surely, every integer is a rational number so that we have

$$\mathbf{I} \subset \mathbf{Q}$$

but every rational number is not an integer, For example, $+\frac{3}{5}$, $-\frac{7}{8}$ are rational numbers but not integers.

Absolute Value of a Rational Number

In respect of each of the two rational numbers

$$+200, -200,$$

we shall refer to the number 200 as its **numerical value** or **absolute value.**

Moreover, we write

$$|+200| = |-200| = 200$$

enclosing $+200$, -200 between two vertical bars.

In general if x is any member of **F**,

we write

$$|+x| = x, |-x| = x$$

and say that the absolute value of each of the rational numbers $+x$ and $-x$ is x.

Also we shall write $|0| = 0$, to that the absolute value of 0 is 0 itself.

For example, we have

$$\left|-\frac{7}{12}\right| = \frac{7}{12}, |-5| = 5, \left|-\frac{14}{9}\right| = \frac{14}{9}, |-1.5| = 1.5$$

$$\left|+\frac{7}{12}\right| = \frac{7}{12}, |+5| = 5, \left|+\frac{14}{9}\right| = \frac{14}{9}, |+1.5| = 1.5,$$

EXERCISE

Put down a few rational numbers and the absolute value of each of them.

Note. Often we shall refer to

$$u$$

as a rational number so that *apparently* neither of signs + nor — is prefixed to it It should be understood in this case that u does not belong to F and that it stands for the compound symbol such as

$$+\frac{3}{4}, -\frac{8}{11}, -\frac{9}{14}, +7$$

The absolute value of u is of course denoted by $|u|$.

Thus

$$u = +\frac{3}{4} \Rightarrow |u| = \frac{3}{4}$$

$$u = -\frac{8}{11} \Rightarrow |u| = \frac{8}{11}$$

$$u = -\frac{9}{14} \Rightarrow |u| = \frac{9}{14}$$

$$u = +7 \Rightarrow |u| = 7$$

34. ADDITION OF RATIONAL NUMBERS

Before giving the *formal definition* of the sum of two rational numbers, we examine the situation in respect of, say, profit and loss. This will provide suitable suggestions for giving the general definition of the sum of two rational numbers.

For example, we consider the following :

(*i*) $(+200)+(+300)$ (*ii*) $(-200)+(-300)$

(*iii*) $(+200)+(-300)$ (*iv*) $(-200)+(+300)$.

In the case of (*i*), both the rational numbers are positive and in the case of (*ii*) they are both negative.

In the cases of (*iii*) and (*iv*), of the two numbers, one is positive and the other negative. While in the case of (*iii*), the absolute value 300 of the negative number -300 is greater than the absolute value 200 of the positive number $+200$, in the case of (*iv*), the absolute value 300 of the positive number $+300$ is greater than the absolute value 200 of the negative number -200.

Now, interpreting the positive number $+x$ as denoting a profit of x rupees and the negative number, $-x$, as denoting a loss of x rupees, we have the following conclusions :

$$(+200)+(+300)=+500 \quad \ldots (i)$$
$$(-200)+(-300)=-500 \quad \ldots (ii)$$
$$(+200)+(-300)=-100 \quad \ldots (iii)$$
$$(-200)+(+300)=+100 \quad \ldots (iv)$$

Let us now try to examine the ideas underlying the above equalities.

In the case of (*i*), the numbers being both positive, the sum is also positive and the absolute value of the sum is the sum of the absolute values.

We have

$$|+200|=200,\ |+300|=300$$
$$200+300=500$$

so that the sum of the absolute values of the two positive numbers is 500.

This gives

$$(+200)+(+300)=+500.$$

In the case of (*ii*), the numbers being both negative, the sum is also negative and the absolute value of the sum is the sum of their absolute values.

Thus, we have

$$(-200)+(-300)=-500.$$

In the case of (*iii*), while one of the two numbers is positive and the other negative, *the absolute value of the negative number is greater than the absolute value of the positive number*. In this case the sum is a negative number whose absolute value

is obtained on subtracting the absolute value of the positive number from that of the negative number so that we have

$$(+200)+(-300) = -(\,|-300| - |+200|\,)$$
$$= -(300-200) = -100.$$

Finally in the case of (*iv*), while one number is positive and the other negative, *the absolute value of the positive number is greater than that of the negative number*. In this case the sum is a positive number whose absolute value is obtained on subtracting the absolute value of the negative number from that of the positive number so that we have

$$(-200)+(+300) = +(300-200) = +100.$$

Before proceeding to give the general definition of the sum of two rational numbers, the reader will find it useful to compute the sums of a few rational numbers in terms of the suggestions outlined above.

We give below a few sums which the reader should do.

(*i*) $(+600)+(-700)$ (*ii*) $(+700)+(-600)$

(*iii*) $(-700)+(+600)$ (*iv*) $(-600)+(+700)$

(*v*) $(+800)+(+600)$ (*vi*) $(-800)+(-600)$

(*vii*) $(+600)+(+800)$ (*viii*) $(-600)+(-800)$

(*ix*) $(+3)+(-4)$ (*x*) $(-3)+(+4)$

(*xi*) $(-4)+(+3)$ (*xii*) $(+4)+(-3)$

(*xiii*) $\left(-\frac{7}{12}\right)+\left(+\frac{3}{2}\right)$ (*xiv*) $\left(+\frac{8}{17}\right)+\left(-\frac{9}{15}\right)$

(*xv*) $\left(+\frac{3}{4}\right)+\left(+\frac{7}{8}\right)$ (*xvi*) $\left(-\frac{3}{4}\right)+\left(-\frac{7}{8}\right)$

(*xvii*) $[(+6)+(-7)]+(-4)$ (*xviii*) $(+6)+[(-7)+(-4)]$.

(*xix*) $\left[\left(-\frac{2}{3}\right)+\left(+\frac{3}{4}\right)\right]+(+2)$

(*xx*) $\left(-\frac{2}{3}\right)+\left[\left(+\frac{3}{4}\right)+(+2)\right]$.

We now give the formal and general definition of the sum of two rational numbers. While defining the sum of two rational numbers, several cases have to be taken care of We consider these one by one.

Sum of Two Rational Numbers

Definition. *In the following* $x, y \in \mathbf{F}$.

(*i*) *Both numbers are positive.*

$$(+x)+(+y) = +(x+y).$$

(*ii*) *Both numbers are negative.*

$$(-x)+(-y) = -(x+y).$$

(*iii*) *One number is positive, and the other negative and the absolute value of the positive number is greater than that of the negative number.*

$$(+x) + (-y) = +(x-y),\ x > y.$$

(*iv*) *One number is positive and the other negative and the absolute value of the negative number is greater than that of the positive number.*

$$(+x) + (-y) = -(y-x),\ y > x.$$

(*v*) *One number is positive and the other negative and the two numbers have the same absolute value.*

$$(+x) + (-x) = 0.$$

(*vi*) *When one or both of the numbers is 0,*

$$(+x) + 0 = +x$$
$$(-x) + 0 = -x$$
$$0 + 0 = 0.$$

Note. It is very important to note that

(*i*) *the sum of two positive rational numbers is positive, and*

(*ii*) *the sum of two negative rational numbers is negative*

EXERCISES

1. Compute $u + v$ and $v + u$ given that

(*i*) $u = (-12),\ v = (+17)$ (*ii*) $u = (-35),\ v = (+12)$

(*iii*) $u = \left(-2\frac{3}{4}\right),\ v = \left(+3\frac{5}{7}\right)$ (*iv*) $u = \left(-\frac{7}{8}\right),\ v = \left(-\frac{3}{5}\right)$

(*v*) $u = (-4{\cdot}45),\ v = (-7{\cdot}35)$ (*vi*) $u = (+14{\cdot}35),\ v = (-12{\cdot}29)$

(*vii*) $u = (-0{\cdot}51),\ v = (+3{\cdot}41)$ (*viii*) $u = (-5),\ v = (+5)$

(*ix*) $u = 0,\ v = (-1{\cdot}4)$ (*x*) $u = -5,\ v = 0$

(*xi*) $u = +\frac{2}{3},\ v = -\frac{2}{3}$ (*xii*) $u = +\frac{7}{3},\ v = 0$

(*xiii*) $u = 0,\ v = \frac{-8}{9}$ (*xiv*) $u = -\frac{3}{5},\ v = +\frac{2}{3}$

2. Compute $(u + v) + w$ and $u + (v + w)$, given that

(*i*) $u = (-9),\ v = (+8),\ w = (-5)$

(*ii*) $u = \left(-\frac{3}{4}\right),\ v = \left(+\frac{5}{12}\right),\ w = \left(-\frac{7}{6}\right)$

(*iii*) $u = (+8{\cdot}25),\ v = (-4{\cdot}35),\ w = (-12{\cdot}75)$

(*iv*) $u = \left(-\frac{2}{3}\right),\ v = \left(+\frac{4}{3}\right),\ w = \left(-\frac{1}{2}\right)$

3. Compute $(u + v) + (w + t)$ and $[(u + v) + w] + t$ given that

(i) $u = (-2), v = (+5), w = (-35), t = (-8)$

(ii) $u = (+2{\cdot}25), v = (-4{\cdot}25), w = (-3{\cdot}35), t = (+7{\cdot}15)$

(iii) $u = \left(-\frac{2}{3}\right), v = \left(+\frac{4}{3}\right), w = \left(-\frac{1}{2}\right), t = \left(+\frac{3}{4}\right)$.

4. Show that

$$|u + v| \leqslant |u| + |v| \quad \forall\, u, v \in \mathbf{Q}.$$

Also illustrate this result by considering a few specific pairs of rational numbers u, v.

Specially give the pairs of numbers u, v such that

$$|u + v| < |u| + |v|.$$

Properties of Addition Composition in **Q**.

I. *Addition is commutative i.e.,*

$$u + v = v + u \;\forall\, u, v \in \mathbf{Q}.$$

The result is an immediate consequence of the definition of the sum of two rational numbers.

II. *Addition is associative i.e.,*

$$u + (v + w) = (u + v) + w \;\forall\, u, v, w \in \mathbf{Q}.$$

Before proceeding with the proof, we consider a specific case.

Let $\qquad u = +5, v = -3, w = -17.$

We have

$$u + v = (+5) + (-3) = +(5 - 3) = +2$$
$$(u + v) + w = (+2) + (-17) = -(17 - 2) = -15.$$

Again,

$$v + w = (-3) + (-17) = -(3 + 17) = -20$$
$$u + (v + w) = (+5) + (-20) = -(20 - 5) = -15.$$

It follows that

$$(u + v) + w = u + (v + w) \text{ in the given case.}$$

In respect of the proof of the associativity of the addition composition in **Q**, we have to consider a number of cases. Of these we shall take up only a few.

(i) *u, v, w are all positive.*

Let $\qquad u = +x, v = +y, w = +z\,;\; x, y, z \in \mathbf{F}.$

We have

$$\begin{aligned}(u + v) + w &= [(+x) + (+y)] + (+z)\\ &= [+(x + y)] + (+z) = +[(x + y) + z],\\ u + (v + w) &= (+x) + [(+y) + (+z)]\\ &= (+x) + [+(y + z)] = +[x + (y + z)].\end{aligned}$$

Now, addition in **F** is known to be associative and $x, y, z \in \mathbf{F}$ so that

$$(x + y) + z = x + (y + z).$$

It follows that

$$(u + v) + w = u + (v + w)$$

(ii) u, v, w are all negative.

Let $u = -x,\ v = -y,\ w = -z\ ;\ x, y, z \in \mathbf{F}$

We have

$$\begin{aligned}(u + v) + w &= [(-x) + (-y)] + (-z)\\ &= [-(x + y) + (-z)] = -[(x + y) + z],\\ u + (v + w) &= (-x) + [(-y) + (-z)]\\ &= (-x) + [-(y + z)]\\ &= -[x + (y + z)]\\ &= -[(x + y) + z] = (u + v) + w.\end{aligned}$$

(iii) u positive, v positive, w negative and

$$|u| + |v| < |w|.$$

Let $u = +x,\ v = +y,\ w = -z$

so that

$$x + y < z.$$

Now

$$x + y < z \Rightarrow y < z - x.$$

Also

$$x + y < z \Rightarrow x < z - y.$$

We have

$$\begin{aligned}(u + v) + w &= [(+x) + (+y)] + (-z)\\ &= [+(x + y)] + (-z)\\ &= -[z - (x + y)].\\ u + (v + w) &= (+x) + [(+y) + (-z)]\\ &= (+x) + [-(z - y)]\\ &= -[(z - y) - x] \doteq -[z - (y + x)]\\ &= -[z - (x + y)]\\ &= (u + v) + w.\end{aligned}$$

The other cases could be similarly disposed of.

Existence of Additive Identity. We have

$$u + 0 = u = 0 + u \ \forall\ u \in \mathbf{Q}.$$

Because of this property, the number, 0, is also referred to as the *Additive Identity* or the neutral element for addition.

Opposite of a Rational Number.

Consider the rational number $+3$. To this rational number corresponds the rational number, -3, such that their sum is the additive identity zero. In fact,

to each rational number corresponds a rational number such that their sum is 0. Thus to $\left(-\frac{11}{17}\right)$ corresponds $\left(+\frac{11}{17}\right)$ and to $\left(+\frac{11}{17}\right)$ corresponds $\left(-\frac{11}{17}\right)$.

In general, to the positive rational number $(+x)$ corresponds the negative rational number $(-x)$ and to the negative rational number $(-x)$ corresponds the positive rational number $(+x)$ such that the sum of the two is the additive identity zero

$$(+x)+(-x)=0.$$

Of course to the rational number 0, corresponds the rational number 0 itself such that

$$0+0=0.$$

Thus, to each rational number u, corresponds a rational number v such that

$$u+v=0=v+u.$$

As an illustration

if $\quad u=+7,$

we have $\quad v=-7$; and

if $\quad u=-\frac{3}{7},$

we have $\quad v=+\frac{3}{7}.$

Each of u, v is called the *additive inverse*, the *Opposite* or the *Negative* of the other and we write

$$u=-v \text{ and } v=-u.$$

Thus, the opposites of the rational numbers

$$\frac{-7}{5},\ +11,\ +\frac{8}{9},\ -\frac{12}{17},\ -\frac{18}{29},\ 0$$

are

$$+\frac{7}{5},\ -11,\ -\frac{8}{9},\ +\frac{12}{17},\ +\frac{18}{29},\ 0$$

respectively

We must distinguish between (*i*) the negative of a number and (*ii*) a nagative number. The negative of a number may not be a negative number and in fact *the negative of a number is positive or negative according as the number is negative or positive*

We shall, in general, use the word opposite of a rational number instead of the negative of a number.

We remark that the opposite of *the opposite of a rational number is the number itself.* Thus, we have

$$-(-u)=u \ \forall\ u \in \mathbf{Q}.$$

We now state and prove a result concerning the opposite of the sum of two rational numbers.

Theorem. *The opposite of the sum of two rational numbers is the sum of their opposites i.e.,*

$$-(u+v)=(-u)+(-v) \ \forall\ u, v \in \mathbf{Q}.$$

Before giving the general proof, we consider a specific case.

Let $$u=-7,\ v=+5$$

so that

$$u+v=(-7)+(+5)=-(7-5)=-2.$$

We have

$$-u=-(-7)=+7$$
$$-v=-(+5)=-5$$
$$\begin{aligned}(-u)+(-v)&=(+7)+(-5)\\&=+(7-5)\\&=+2=-(-2)\\&=-(u+v).\end{aligned}$$

Proof

$$\begin{aligned}&(u+v)+[(-u)+(-v)]\\&=(v+u)+[(-u)+(-v)]\\&=v+\{u+[(-u)+(-v)]\}\\&=v+\{[u+(-u)]+(-v)\}\\&=v+\{0+(-v)\}=v+(-v)=0.\end{aligned}$$

Finally

$$\begin{aligned}&(u+v)+[(-u)+(-v)]=0\\&\Rightarrow -(u+v)=(-u)+(-v).\end{aligned}$$

EXERCISES

1. Give the opposites of the following rational numbers :

(*i*) $+3$ (*ii*) $\frac{-3}{3}$

(*iii*) $-2{\cdot}25$ (*iv*) $(+3)+(-5)$

(*v*) $\left(-\frac{2}{3}\right)+\left(-\frac{3}{4}\right)$ (*vi*) $(+3)+\left(-\frac{2}{3}\right)$.

2. Put down any five positive and any five negative rational numbers and give their opposites.

3. Add -3 to the negative of -3 and $+4$ to the negative of $+4$.

4. Verify the result of the above theorem by taking any five pairs u, v of rational numbers.

Note 1. It is important to note that the absolute values of a rational number and its opposite are equal i.e., $|u| = |-u| \ \forall \ u \in Q$.

In particular $\quad |-3| = |-(-3)|$.

2. It will be seen that in the above discussion we have used the *minus* sign, $-$, in two different senses. Placed before any fraction, it denotes a negative rational number and placed before a rational number, it denotes the opposite of the same.

Considering the fractions

$$3, \frac{3}{5}, \frac{5}{7}$$

we see that

$$-3, -\frac{3}{5}, -\frac{5}{7}$$

are negative rational numbers. Again, considering the rational number

$$-3, \frac{-3}{5}, \frac{-5}{7}, +2, +4$$

we see that

$$-(-3), -\left(\frac{-3}{5}\right), -\left(\frac{-5}{7}\right), -(+2), -(+4)$$

denote the rational numbers

$$+3, \frac{+3}{5}, \frac{+5}{7}, -2, -4.$$

There is also a third use of the minus sign, $-$, to indicate subtraction in which case it is sandwitched between two numbers and not prefixed to a number. We must, therefore, avoid confusion between these *three* uses of the minus sign.

Subtraction

Consider any two rational numbers u, v. Then the expression

$$u - v$$

denotes the rational number w, if it exists, such that

$$u = v + w.$$

We then write

$$w = u - v \Leftrightarrow u = v + w.$$

We shall show that w exists corresponding to *each* pair of rational numbers u, v.

Firstly, we shall consider some specific cases.

1. Let $\quad u = +3, \ v = -7,$

so that we consider

$$(+3) - (-7).$$

We seek a number w such that

$$w + (-7) = +3.$$

A little thought will suggest that

$$w = (+10)$$

will do the job.

Again, consider

$$(-7) - (+5).$$

We seek a number w such that

$$w + (+5) = -7$$

We may see that

$$w = -12.$$

We now state and prove a theorem.

Theorem.

$$u - v = u + (-v) \ \forall\ u, v \in \mathbf{Q}.$$

Proof. We have

$$[u + (-v)] + v = u + [(-v) + v] = u + 0 = u$$
$$\Rightarrow u + (-v) = u - v.$$

Rule. *To subtract v from u add to u the opposite of v.*

In symbols

$$u - v = u + (-v), u, v \in \mathbf{Q}.$$

Examples

(*i*) $(+12) - (+3) = (+12) + (-3) = +9$

(*ii*) $(-8) - (-10) = (-8) + (+10) = +2$

(*iii*) $(-5{\cdot}42) - (-6{\cdot}17) = (-5{\cdot}42) + (+6{\cdot}17) = +0{\cdot}75$

(*iv*) $\left[+\frac{9}{11}\right] - \left[-\frac{6}{22}\right] = \left[+\frac{9}{11}\right] + \left[+\frac{6}{22}\right] = +\frac{24}{22} = +\frac{12}{11}$

(*v*) $\left[-4\frac{1}{3}\right] - \left[-2\frac{2}{3}\right] = \left[-4\frac{1}{3}\right] + \left[+2\frac{2}{3}\right] = -\frac{5}{3}$

(*vi*) $\left[-7\frac{1}{2}\right] - \left[+2\frac{2}{5}\right] = \left[-7\frac{1}{2}\right] + \left[-2\frac{2}{5}\right] = -9\frac{9}{10}.$

EXERCISES

1. Compute $u - v$ given that

(*i*) $u = +8, \ v = -2$ (*ii*) $u = -21, v = 0$

(*iii*) $u = 0, \ v = -\frac{2}{3}$ (*iv*) $u = +0{\cdot}25, \ v = +0{\cdot}05$

(*v*) $u = -\frac{5}{6}, \ v = +\frac{10}{12}.$

2. Compute

$$u + (v - w), (u - v) + w, (u - v) - w$$

given that

(i) $u = +7,\ v = -3,\quad w = -5$

(ii) $u = \frac{-7}{3},\ v = +\frac{2}{5},\ w = -\frac{1}{8}$

(iii) $u = -1{\cdot}25,\ v = -2{\cdot}35,\ w = +1{\cdot}05.$

3. Compute $|u - v|$

given that

(i) $u = +3,\ v = -5$ (ii) $u = -\frac{7}{3},\ v = -\frac{2}{5}$

(iii) $u = -1,\ v = +\frac{5}{3}$ (iv) $u = +\frac{3}{5},\ v = +\frac{2}{7}$

Note. Having considered the opposite of the sum of two rational numbers already, we now consider the opposite of the difference of two numbers.

Theorem. $-(u - v) = v - u \quad \forall\ u, v \in Q.$

Proof. We have

$$\begin{aligned}(u - v) + (v - u) &= [u + (-v)] + [v + (-u)] \\ &= u + \{(-v) + [v + (-u)]\} \\ &= u + \{[(-v) + v] + (-u)\} \\ &= u + \{0 + (-u)\} = u + (-u) = 0\end{aligned}$$

so that we have

$$v - u = -(u - v).$$

EXERCISE

Verify the result proved above, taking

(i) $u = (+7),\ v = (-3)$ (ii) $u = -\frac{8}{5},\ v = -\frac{3}{2}$

(iii) $u = +\frac{7}{3},\ v = -\frac{5}{4}$ (iv) $u = -3{\cdot}25,\ v = +1{\cdot}15.$

35. MULTIPLICATION OF RATIONAL NUMBERS

Before proceeding to give the general definition of the product of two rational numbers, we take up a special case and lay down some considerations which will motivate and suggest the general definition.

We consider the products :

(i) $(+3) \times (+2)$
(ii) $(+3) \times (-2)$
(iii) $(-3) \times (-2)$
(iv) $(-3) \times (+2)$.

Consider a car moving with some velocity u. Thus, for instance, the car may be moving towards the east at the rate of 30 kilometres per hour.

$$(+2) \times u$$

denotes the velocity whose magnitude is twice, *i.e.*, $+ \mid 2 \mid$ times that of the velocity u and whose direction is the same as that of u.

Again

$$(-2) \times u$$

denotes the velocity whose magnitude is twice, *i e.*, $\mid -2 \mid$ times that of u but whose direction is opposite of that of u.

We now consider the velocities

(*i*) $(+3) \times (+2) \times u$

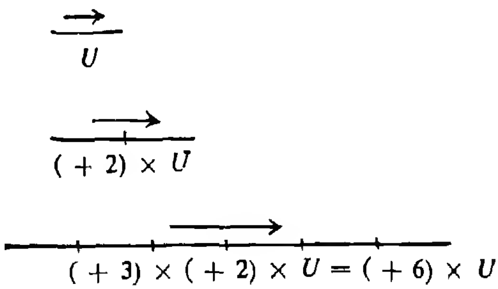
(*ii*) $(+3) \times (-2) \times u$

(*iii*) $(-3) \times (-2) \times u$

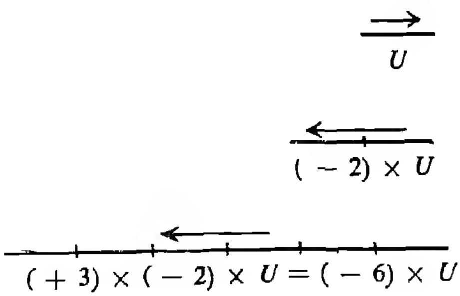
(*iv*) $(-3) \times (+2) \times u$.

The position *vis-a-vis* each of these four cases is shown in the corresponding figures

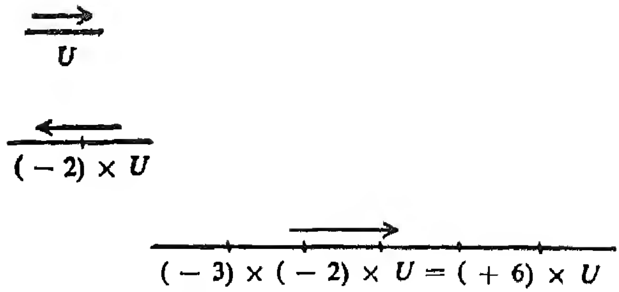
(*i*)



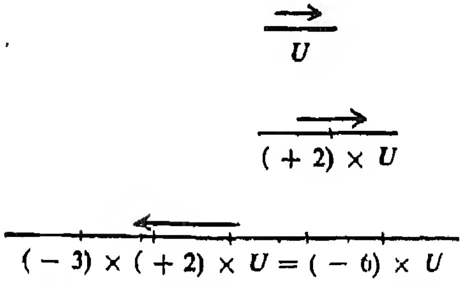
(*ii*)



(*iii*)



(*iv*)



It will thus be seen that whereas the multiplication of the velocity with a positive number keeps the direction of the velocity intact that with a negative number reverses the direction of the velocity. In particular, therefore, successive multiplication with two negative numbers keeps the direction of the velocity intact.

Thus, it will appear that we have the following equalities.

$$(+3) \times (+2) = +(3 \times 2) = (+6)$$
$$(+3) \times (-2) = -(3 \times 2) = (-6)$$
$$(-3) \times (-2) = +(3 \times 2) = (+6)$$
$$(-3) \times (+2) = -(3 \times 2) = (-6)$$

We are now led to give the following *definition* of the product of two numbers. We have, of course, to consider different cases.

I. *u, v are both positive.*

$$u \times v = +(|u| \times |v|)$$

II. *u, v are both negative.*

$$u \times v = +(|u| \times |v|)$$

III. *u is positive, v is negative.*

$$u \times v = -(|u| \times |v|)$$

IV. *u is negative, v is positive.*

$$u \times v = -(|u| \times |v|)$$

V. *One number is 0.* The product is zero, whatever the second number may be.

We may write $u \,.\, v$ or uv in place of $u \times v$.

The rules could as well be stated as follows :

In the following $x, y \in F$.

(*i*) $(+x) \times (+y) = +(x \times y)$
(*ii*) $(-x) \times (-y) = +(x \times y)$
(*iii*) $(+x) \times (-y) = -(x \times y)$
(*iv*) $(-x) \times (+y) = -(x \times y)$.

Note. It will be seen that the *product of two numbers both positive or negative is positive*. Also the *product of two numbers, one positive and the other negative, is negative.*

EXERCISES

1. Compute the following :

(*i*) $(+25)(-11)$ (*ii*) $\left(-\frac{3}{4}\right)\left(\frac{7}{8}\right)$

(*iii*) $(+1{\cdot}02) \times \left(-\frac{1}{2}\right)$ (*iv*) $(+7{\cdot}6)(-0.8)$

(*v*) $(-0{\cdot}20)(+5)$ (*vi*) $\left(-\frac{3}{10}\right)\left(+\frac{5}{7}\right)$.

2. Compute $(u \times v) \times w$, $u \times (v \times w)$ given that

(*i*) $u = -3$, $v = +5$, $w = +8$
(*ii*) $u = -\frac{3}{4}$, $v = +\frac{4}{5}$, $w = +\frac{2}{3}$
(*iii*) $u = -0{\cdot}7$, $v = -24$, $w = 0$
(*iv*) $u = -3{\cdot}5$, $v = -0{\cdot}2$, $w = +6$
(*v*) $u = +3$, $v = -4$, $w = -1{\cdot}5$.

3. Compute the additive inverses, *i.e.*, the opposites of the following products.

(*i*) $(-7)(-3)$ (*ii*) $(-2)(+9)$

(*iii*) $\left(-\frac{4}{3}\right)\left(+\frac{6}{5}\right)$ (*iv*) $\left(+\frac{3}{7}\right)\left(+\frac{2}{3}\right)$.

4. Compute $(u - v)(w - t)$ given that

(*i*) $u = +\frac{1}{2}$, $v = -\frac{1}{3}$, $w = -\frac{1}{5}$, $t = 0$
(*ii*) $u = -6$, $v = +2$, $w = +2$, $t = -3$
(*iii*) $u = -4$, $v = -5$, $w = +1$, $t = -2$

5. Fill in the blanks to make the following statements true.

(*i*) $+ 5 \times$ ——— $= 30$ (*ii*) $+ 3 \times$ ——— $= - 6$

(*iii*) $- 3 \times$ ——— $= + 9$ (*iv*) $- 8 \times$ ——— $= 4$

(*v*) $- 4 \times$ ——— $= + 1$ (*vi*) $+ \frac{3}{7} \times$ ——— $= + 1.$

Properties of Multiplication Composition in **Q.**

1. *Multiplication composition in* **Q** *is commutative, i.e.,*

$$u \times v = v \times u \ \forall\ u, v \in \mathbf{Q}.$$

The truth of this statement is an immediate consequence of the commutativity of multiplication in **F** which guaranties that

$$|u| \times |v| = |v| \times |u|, \ |u|, |v| \in \mathbf{F}.$$

II. *Multiplication composition in* **Q** *is associative, i.e.,*

$$(u \times v) \times w = u \times (v \times w) \ \forall\ u, v, w \in \mathbf{Q}.$$

Firstly we consider a specific case.

Let

$$u = -3, \quad v = +6, \quad w = -5.$$

We have

$$\begin{aligned} u \times v &= (-3) \times (+6) = (-3 \times 6) = -18 \\ (u \times v) \times w &= (-18) \times (-5) = +(18 \times 5) = +90 \\ v \times w &= (+6) \times (-5) = -(6 \times 5) = -33 \\ u \times (v \times w) &= (-3) \times (-30) = +(3 \times 30) = +90 \end{aligned}$$

so that it follows that

$$(u \times v) \times w = u \times (v \times w)$$

for the given rational numbers u, v, w.

Now consider the general case. Let u, v, w be any three rational numbers.

It may be easily seen that

$$\begin{aligned} (u \times v) \times w &= +[(|u| \times |v|) \times |w|] \\ u \times (v \times w) &= +[|u| \times (|v| \times |w|)] \end{aligned}$$

if of u, v, w, all the numbers are positive or if two are negative and one positive.

Also because of the associativity of multiplication composition in **F** to which $|u|, |v|, |w|$ all belong, we have

$$(|u| \times |v|) \times |w| = |u| \times (|v| \times |w|).$$

It follows that we have

$$(u \times v) \times w = u \times (v \times w)$$

in this case.

The truth of this result may be seen to be true if u, v, w are all negative or if two are positive and one is negative. In this case, we shall have

$$(u \times v) \times w = -(|u| \times |v| \times |w|)$$
$$= u \times (v \times w).$$

Multiplicative Identity

Multiplicative property of the rational number $+1$.

Theorem.

$$u \times (+1) = u \;\forall\; u \in \mathbf{Q}.$$

Proof.

Case I. u is positive, so that we have

$$u = +|u|.$$

By the definition of product,

$$u \times (+1) = +(|u| \times 1) = +|u| = u.$$

Case II. u is negative so that we have

$$u = -|u|.$$

By the definition of product,

$$u \times (+1) = -(|u| \times 1)$$
$$= -|u| = u.$$

Case III. u is zero, so that we have

$$u \times (+1) = 0 \times (+1) = 0 = u.$$

Multiplicative Inverses or Reciprocals of Non-zero Rational Numbers

Theorem. To each non-zero rational number u there corresponds the non-zero rational number v such that

$$u \times v = +1.$$

Proof.

Case I. u is positive so that we have

$$u = |u|.$$

Surely $|u| \in \mathbf{F}$. We consider the rational number v defined as follows

$$v = +\frac{1}{|u|}.$$

We have u, v being both positive

$$u \times v = +\left(|u| \times \frac{1}{|u|}\right) = +1.$$

Case II. u is negative so that we have

$$u = -|u| \quad , \quad |u| \in \mathbf{F}.$$

We take

$$v = -\frac{1}{|u|}.$$

We have u, v being both negative

$$u \times v = + \left(|u| \times \frac{1}{|u|} \right) = +1.$$

Definition. *The non-zero rational number v corresponding to the non-zero rational number u such that $u \times v = +1$* is called the *reciprocal of u.*

In fact each of u, v is the reciprocal of the other.

Essentially each of u, v is the *multiplicative inverse* of the other.

Note. It is very important to notice that *the rational number zero does not admit of any reciprocal.*

It is because

$$u \times 0 = 0 \qquad \forall\, u \in \mathbf{Q}.$$

Let us examine the situation *vis-a-vis* the existence of the reciprocal of zero.

Let, if possible, 0 admit of the reciprocal u so that we have

$$u \times 0 = 1.$$

Also we have

$$u \times 0 = 0.$$

Thus

$$\left.\begin{array}{l} u \times 0 = 1 \\ u \times 0 = 0 \end{array}\right\} \Rightarrow 0 = 1.$$

We see that on the basis of the assumption of 0 admitting of the reciprocal, we arrived at a *false statement* $0 = 1$. It follows that '0' cannot be regarded as admiting of a reciprocal.

Positively speaking, we may note that the theorem has demonstrated the existence of the reciprocal of every non-zero rational number.

We now demonstrate the converse of the above result and show that

$$uv = 0 \Rightarrow u = 0 \text{ and/or } v = 0.$$

Let u, v be two rational numbers such that

$$uv = 0.$$

Let us suppose that $u \neq 0$.

Now u, being non-zero rational, admits of the reciprocal, say w

We have

$$\begin{aligned} & uv = 0 \\ \Rightarrow\ & w(uv) = w \times 0 \\ \Rightarrow\ & (wu)\, v = 0 \\ \Rightarrow\ & (+1)\, v = 0 \\ \Rightarrow\ & v = 0. \end{aligned}$$

Thus the assumption of, u, being non-zero leads to the conclusion that $v = 0$.

We may similarly show that assumption of v being non-zero will lead to the conclusion that $u = 0$.

Thus we have proved the result as stated.

In fact we have

$$uv = 0 \Leftrightarrow u = 0 \text{ and/or } v = 0.$$

EXERCISE

Give the reciprocals of each of the following non-zero rational numbers.

(*i*) -3. (*ii*) $-\frac{2}{3}$ (*iii*) $+\frac{7}{8}$

(*iv*) $+2{\cdot}32$ (*v*) $-3{\cdot}25$ (*vi*) $-0{\cdot}35$

(*vii*) $-\frac{5}{4}$ (*viii*) $+\frac{4}{5}$ (*ix*) $-7{\cdot}05$

(*x*) $(-3)+(-5)$ (*xi*) $(+3)+(-2)$

(*xii*) $(+4)-(+5)$ (*xiii*) $\left(-\frac{2}{3}\right)\times\left(+\frac{4}{5}\right)$

(*xiv*) $\left(+\frac{1}{4}\right)\times\left(-\frac{7}{6}\right)$ (*xv*) $\left(-\frac{1}{2}\right)\times(-{\cdot}3)$.

Division

Let u, v be two rational numbers. We suppose that v is a non-zero rational number.

We now seek to give a meaning to the expression.

$$u \div v.$$

Now $u \div v$ denotes the number w, *if it exists*, such that

$$u = vw.$$

Thus, we have

$$u \div v = w \Leftrightarrow u = vw.$$

Before considering the general case, we consider some specific cases :

(*i*) $(+6)\div(-2) = -3$ for $(-2)\times(-3) = +6$

(*ii*) $(+5)\div(-3) = -\frac{5}{3}$ for $(-3)\times\left(-\frac{5}{3}\right) = +5$

(*iii*) $\left(-\frac{7}{8}\right)\div\left(-\frac{3}{4}\right) = +\frac{7}{6}$ for $\left(-\frac{3}{4}\right)\times\left(+\frac{7}{6}\right) = -\frac{7}{8}$.

Now, in general, we have $x, y \in \mathrm{F}$

$$(+x)\div(+y) = +(x \div y)$$
$$(-x)\div(-y) = +(x \div y)$$
$$(+x)\div(-y) = -(x \div y)$$
$$(-x)\div(+y) = -(x \div y).$$

Also, we notice that

$$u \div v$$

is positive if the numbers u, v are both positive or both negative and $u \div v$ is negative if one of the numbers u, v is positive and the other negative.

Reciprocal of a Non-zero Rational Number as a Quotient

Let v be any non-zero rational number. We show that the reciprocal of v is the rational number

$$(+1) \div v.$$

We have

$$(+1) \div v = w \Rightarrow v \times w = +1$$

so that w is the reciprocal of v.

It follows that the reciprocal of the non-zero rational number v is the rational number

$$(+1) \div v.$$

As a result of this expression for the reciprocal of a non-zero rational number, we have

$$u \div v = u \times (+1 \div v)$$

so that

$$u \div v$$

is the product of u with the reciprocal of v.

Instead of $u \div v$, we also often use the alternative symbols

$$\frac{u}{v} \text{ or } u/v.$$

Note 1. It should be noticed that

$$u \div v$$

has a meaning only for non-zero rational number v. Thus, we remark that division by 0 is a meaningless operation

Note 2. It will be found useful to have a separate name for the set of non-zero rational numbers and as such we shall denote this set by

$$Q_0.$$

Thus, the set Q_0 differs from the set Q only in respect of the number zero so that zero is the only number which belongs to Q but not to Q_0. Of course, we have

$$Q_0 \subset Q.$$

Reciprocal of the Product of Two Non-zero Rational Numbers

Theorem. *Reciprocal of the product of two non-zero rational numbers is the product of their reciprocals.*

Proof. Let u, w be two non-zero rational numbers and let v, t be their reciprocals respectively.

We have

$$uv = +1$$
$$wt = +1.$$

Thus we have

$$(uv)(wt) = (+1)(+1) = +1.$$

Because of the commutativity and associativity of the multiplication composition, in **Q**, we have

$$+1 = (uv)(wt) = (uw)(vt)$$

showing that

$$vt \text{ is the reciprocal of } uw$$

i.e.,

$$\begin{aligned} +1 \div (uw) &= vt \\ &= (+1 \div u)(+1 \div w) \end{aligned}$$

using the alternative expression, we have

$$\frac{+1}{uw} = \frac{+1}{u} \cdot \frac{+1}{w}.$$

As illustrations, we see that

$$\frac{+1}{(+2)(-3)} = \frac{+1}{+2} \cdot \frac{+1}{-3} = \left(+\frac{1}{2}\right)\left(-\frac{1}{3}\right) = -\frac{1}{6}$$

$$\frac{+1}{(-4)(-5)} = \frac{+1}{-4} \cdot \frac{+1}{-5} = \left(-\frac{1}{4}\right)\left(-\frac{1}{5}\right) = +\frac{1}{20}.$$

Distributive Law

Theorem.

$$u(v + w) = uv + uw \;\forall\; u, v, w \in \mathbf{Q}$$

Let us first consider a specific case.

Taking

$$u = -3, \quad v = -5, \quad w = +2,$$

we have

$$\begin{aligned} v + w &= (-5) + (+2) = -3 \\ u(v + w) &= (-3)(-3) \quad = +9 \\ uv &= (-3)(-5) \quad = +15 \\ uw &= (-3)(+2) \quad = -6 \\ uv + uw &= (+15) + (-6) = +9. \end{aligned}$$

It follows that

$$u(v + w) = uv + uw$$

when u, v, w have the values as given above.

To prove the general case, we have to consider several cases.

We consider only one of them when u is positive and v, w are both negative.

Let

$$u = +x,\quad v = -y,\quad w = -z.$$

We have

$$\begin{aligned} v + w &= -(y+z) \\ u(v+w) &= (+x)[-(y+z)] \\ &= -[x(y+z)] = -(xy+xz) \\ uv &= -(xy) \\ uw &= -(xz). \end{aligned}$$

Also

$$\begin{aligned} uv + uw &= -(xy+xz) \\ &= u(v+w). \end{aligned}$$

The other cases may be similarly disposed of.

36. THE RELATION 'IS GREATER THAN' IN THE SET OF RATIONAL NUMBERS.

Definition. *If $u, v \in \mathbf{Q}$, we say that*

u is greater than v

if

$u - v$ is positive.

We write symbolically

$$u > v$$

to denote that u is greater than v.

Thus, we have

$$u > v \Leftrightarrow u - v \text{ is positive.}$$

Also we have

v is less than $u \Leftrightarrow u$ is greater than v or symbolically

$$v < u \Leftrightarrow u > v.$$

Illustrations

(*i*) $+7 > +5$ because $(+7)-(+5) = (+7)+(-5) = +2$
(*ii*) $+5 > -3$ because $(+5)-(-3) = (+5)+(+3) = +8$
(*iii*) $-7 > -9$ because $(-7)-(-9) = (-7)+(+9) = +2.$

It is important to notice that

$$+x > +y \Leftrightarrow x > y$$
$$-x > -y \Leftrightarrow x < y.$$

It is very important to remember that a negative rational number is greater than another negative rational number if and only if its absolute value is smaller than the other. Thus, for example,

$$-13 > -17 \text{ because } |-13| < |-17|$$

$$-\frac{3}{4} > -2 \text{ because } \left|-\frac{3}{4}\right| < |-2|.$$

In general

$$-x > -y \Leftrightarrow |-x| < |-y|, x, y \in F.$$

Also every positive rational number is greater than every negative rational number *i.e.*,

$$+x > -y$$

x, y being any fractions whatsoever.

As illustrations, we see that

$$+\frac{9}{8} > +\frac{7}{9}$$
$$-\frac{7}{9} > -\frac{9}{8}$$
$$+\frac{9}{8} > -\frac{7}{9}$$
$$+\frac{7}{9} > -\frac{9}{8}.$$

Also we may notice that every positive number is greater than 0 and the number 0 is greater than every negative number, *i.e.*, we have

$$+x > 0 > -y$$

whatever the fractions x, y may be.

In fact, we have

$$(+x) - 0 = +x$$
$$0 - (-y) = +y.$$

As illustration, we have

$$+\frac{7}{3} > 0 > -\frac{2}{5}.$$

EXERCISES

1. Arrange the following in ascending order.

(*i*) $-7, -\frac{11}{13}, +0{\cdot}25, +3, 0, -17, -9, +8$

(*ii*) $-3, +\frac{9}{3}, +\frac{1}{4}, -\frac{7}{12}, +\frac{5}{6}$

(*iii*) $+\frac{3}{4}, +\frac{1}{4}, 0, -3, +10, -0{\cdot}25$

(*iv*) $-\frac{1}{3}, +\frac{3}{4}, -\frac{1}{6}, +\frac{1}{2}, -\frac{1}{4}, +\frac{5}{6}$

(*v*) $+\frac{8}{12}, -3, -\frac{2}{3}, +2\frac{1}{3}$

(*vi*) $+2, 0, -21, -7, +12, +21, -6, -12.$

2. Which of the following statements are true ?

(*i*) $-\cdot 3 > +\cdot 03$ (*ii*) $+\cdot 10 > +\cdot 02$

(*iii*) $+1\cdot 37 < +1\cdot 378 < +1\cdot 38$.

Properties of the Relation 'Is greater than'.

Trichotomy Law.

Theorem. *Given any two rational numbers u, v there exists one and only one of the following three possibilities*

(*i*) $u > v$ (*ii*) $v > u$ (*iii*) $u = v$.

Proof. Vis-a-vis a rational number, we have one and only one of the following three possibilities of its being

(*i*) positive (*ii*) negative (*iii*) zero.

Thus, in respect of $u - v$ we have one and only one of the three possibilities

(*i*) $u - v$ is positive. (*ii*) $u - v$ is negative. (*iii*) $u - v$ is zero.

We consider these cases one by one.

(*i*) $u - v$ is positive $\Leftrightarrow u > v$

(*ii*) If $u - v$ is negative, we have

$$-(u - v) = v - u$$

is positive. Also

$$v - u \text{ is positive} \Leftrightarrow v > u$$

(*iii*) $u - v = 0 \quad \Leftrightarrow u = v$.

Transitivity.

Theorem. $u > v$ and $v > w \Rightarrow u > w$.

Proof.

$$u > v \Rightarrow u - v \text{ is positive}$$
$$v > w \Rightarrow v - w \text{ is positive.}$$

Also

$$u - v,\ v - w \text{ being positive,}$$
$$\begin{aligned}(u - v) + (v - w) &= [u + (-v)] + [v + (-w)] \\ &= u + [(-v) + v] + (-w) \\ &= u + 0 + (-w) \\ &= u + (-w) = u - w\end{aligned}$$

is positive, implying that

$$u > w.$$

Compatibility with the Addition Composition.

Theorem. $u > v \Rightarrow u + w > v + w$.

Proof.

$$u > v \Rightarrow u - v \text{ is positive.}$$

Also

$$\begin{aligned}(u+w)-(v+w) &= (u+w)+[-(v+w)] \\ &= (u+w)+[(-v)+(-w)] \\ &= (u+w)+[(-w)+(-v)] \\ &= u+[w+(-w)+(-v)] \\ &= u+0+(-v)=u-v.\end{aligned}$$

Thus, we see that

$$(u+w)-(v+w)$$

is positive, implying

$$u+w>v+w.$$

Compatibility with the Multiplication Composition.

Theorem.

$$u>v,\ w>0 \Rightarrow uw>vw.$$

Proof.

$$u>v \Rightarrow u-v \text{ is positive.}$$

Also $u-v$, w being both positive, we have

$$(u-v)\,w = u\,w - v\,w$$

is positive, implying that

$$u\,w > v\,w.$$

Cor. $\quad u>v,\ w<0 \Rightarrow u\,w < v\,w.$

37. USUAL NOTATION FOR POSITIVE RATIONAL NUMBERS.

Let

$$\frac{a}{b}$$

be any fraction so that $a, b \in \mathrm{N}$.

The two rational numbers corresponding to this fraction are

$$+\frac{a}{b},\ -\frac{a}{b}.$$

We now decide to *drop* the sign, $+$, before positive rational numbers so that we agree to regard

$$\frac{a}{b},\ a \in \mathrm{N},\ b \in \mathrm{N}$$

itself as a positive rational number

From this point each fraction will be hereafter thought of as a positive rational number.

Thus, for example

$$3,\ \frac{5}{3},\ 2\ 35$$

will be thought of as identified with the positive rational numbers

$$+3, +\frac{5}{3}, +2{\cdot}35.$$

In terms of this understanding, for example,

$$3-4+5-7$$

is the same as

$$(+3)-(+4)+(+5)-(+7).$$

It should be seen that this agreement does not lead to any confusion. For example, the statements

$$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$
$$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$$

are true if we interpret

$$\frac{1}{2}, \frac{1}{3}, \frac{5}{6}, \frac{1}{6}$$

as fractions or as the corresponding positive rational numbers

$$+\frac{1}{2}, +\frac{1}{3}, +\frac{5}{6}, +\frac{1}{6}.$$

This is because of the manner in which we defined the sum and product of two positive rational numbers which we reproduce below :

$$(+x)+(+y)=+(x+y)$$
$$(+x)\times(+y)=+(x\times y).$$

Again, in respect of the relation 'Is greater than', we know that

$$+x>+y \Leftrightarrow x>y.$$

38. INTEGRAL POWERS OF RATIONAL NUMBERS.

The meaning of the symbol

$$x^n\,;\, x\in \mathbf{Q}, n\in \mathbf{I}.$$

The consideration will be split up into three parts.

(*i*) The index n is a positive integer.

(*ii*) The index n is zero.

(*iii*) The index n is a negative integer.

Case I. *Let the index be a positive integer.* By definition

$$x^n=\underbrace{x\times x\times x\times x\times \ldots\ldots\ldots\ldots \times x}_{n\text{-times}},\ x\in \mathbf{Q}$$

Thus, we have

$$x^1 = x$$
$$x^2 = x \times x$$
$$x^3 = x \times x \times x$$
$$x^4 = x \times x \times x \times x$$

and so on.

x^n is read as n-th power of x. Also we often read x^2 as x-squared and x^3 as x-cubed.

The following results are immediate consequences of our definition.

(*i*) $$x^n \times x^m = x^{n+m}$$

(*ii*) If $$x \neq 0$$

$$\frac{x^n}{x^m} = x^{n-m} \text{ if } n > m$$

$$\frac{x^n}{x^n} = 1$$

$$\frac{x^n}{x^m} = \frac{1}{x^{m-n}} \text{ if } m > n.$$

Here m, n are positive integers.

Case II. *Let the index be zero.* We seek a meaning for the symbol x^0.

If we require the result

$$\frac{x^n}{x^m} = x^{n-m},$$

to remain true for $n = m$, we must have

$$1 = x^0, x \neq 0.$$

Thus, we define

$$x^0 = 1,$$

x being any non-zero rational number.

We have given no meaning to the symbol x^0 when x is the rational number zero.

Case III. Let the index be a negative integer. If we require the result $\frac{x^n}{x^m} = x^{n-m}$ to remain true for $m > n$ we see that $\frac{x^0}{x^m} = x^{0-m} = x^{-m}$; m being a positive integer $\Rightarrow x^{-m} = \frac{1}{x^m}$, $x \neq 0$.

Thus we define

$$x^{-m} = \frac{1}{x^m} \ ; x \neq 0.$$

It should be seen that when the index, $-m$, is a negative integer, the symbol

$$x^{-m}$$

has been given a meaning only for non-zero values of x. For example, the symbol

$$x^{-2}$$

has been given no meaning when x is zero.

Illustrations.

$$\left(\frac{1}{2}\right)^{-2} = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1}{4}} = 4$$

$$(-3)^0 = 1$$

$$(-4)^{-3} = \frac{1}{(-4)^3} = \frac{1}{-64} = -\frac{1}{64}.$$

EXERCISE

Which of the following statements is true and justify your answer.

(*i*) $(x^4)(x^{-2}) = x^2 \ \forall \ x \in Q_0$

(*ii*) $(x^3)^2 = x^6 \ \forall \ x \in Q$

(*iii*) $x^{(3^2)} = x^9 \ \forall \ x \in Q.$

(*iv*) $(x^{-5})(x^{-4}) = x^{-9} \ \forall \ x \in Q$

(*v*) $\frac{1}{x^3} \cdot \frac{1}{x^6} \cdot \frac{1}{x^{-2}} = x^{-7} \ \forall \ x \in Q$

(*vi*) $(xy)^{-2}\ (xy) = \frac{1}{xy} \ \forall \ x, y \in Q_0$

(*vii*) $2^{-1}.3^{-2} = \frac{1}{2\,3^2}$

(*viii*) $3 \cdot x^m = \frac{3}{x^{-m}} \ \forall \ x \in Q.$

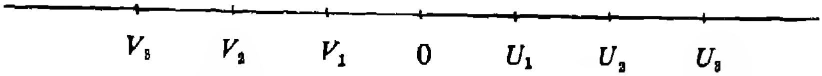
39. REPRESENTATION OF RATIONAL NUMBERS BY POINTS ALONG A LINE.

Consider a line which we suppose as drawn parallel to the printed lines of the page.

We name any point, 0, on the line to be called the *Origin.*

The point 0 divides the line into two parts so that the points, other than 0, lie either to the right or to the left of 0.

The part of the line to the right of the point 0 will be termed the *positive side* and that to the left of 0 the *negative side.*



We take any unit of length and let U_1 be the point on the positive side of 0 such that $0U_1$ is of unit length.

We step out to the right of the 0 distances each equal to the unit length $0U_1$ and suppose that the points thus obtained are denoted by

$$U_1, U_2, U_3, U_4, \dots \dots$$

We say that these points correspond to the positive integers

$$1, 2, 3, 4, \dots$$

respectively.

The point, 0, is said to correspond to the integer zero.

We similarly step out to the left of 0 distances each equal in length to $0U_1$ and obtain points

$$V_1, V_2, V_3, V_4, \dots\dots$$

We say that these points correspond to the negative integers

$$-1, -2, -3, -4, \dots$$

Thus, for example, if a point P corresponds to the positive integer 15, it means that the point P is to the right of 0 and the distance $0P$ is 15 units of length. Also if a point Q corresponds to the negative integer, -15, this means that the point Q is to the left of 0, and the distance $0Q$ is 15 units of length.

Now consider any positive rational number

$$\frac{a}{b},$$

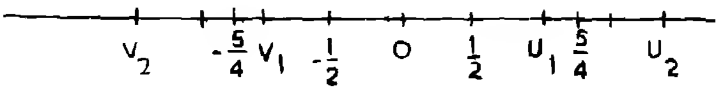
a, b being natural numbers.

We suppose that $0U_1$ is divided into b equal parts and we step out to the right of 0, a times, each step being equal in length to the *b-th* part of the unit lenth $0U_1$. The point, thus obtained, is said to be corresponding to the positive rational number

$$\frac{a}{b} \; ; a, b \in \mathbf{N}.$$

Stepping out a times to the left of 0, we obtain the point corresponding to the negative rational number

$$\frac{a}{b}.$$

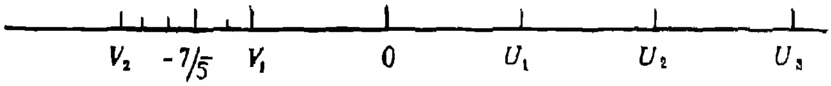


Thus we have learnt to associate to each rational number a point of the line such that the measure of the distance of this point from the origin is equal to the absolute value of the rational number representing the point.

Thus, for example, the measures of the distance from the origin of the point representing the numbers

$$+3, -\frac{7}{5}$$

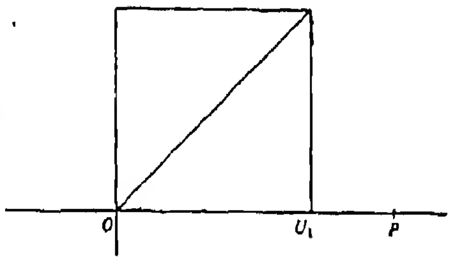
are $3, \frac{7}{5}$ respectively.



Having been able to associate with each rational number a point on the line, the following question now naturally arises.

Shall we in this manner exhaust every point of the line, *i.e.*, will in this process every point of the line be associated to some rational number ?

The answer to this question is the emphatic No. as was first established by Pythagoras some 2,500 years ago. He showed that if P be the point on the line such that OP is equal in length to the diagonal of the square of side equal to unit length $0U_1$, then no rational number will correspond to the point P.



We proceed to show this.

If possible, let $\frac{a}{b}$ be the rational number corresponding to the point P. Then we have

$$1^2 + 1^2 = \left(\frac{a}{b}\right)^2 \Rightarrow a^2 = 2b^2.$$

We consider the natural numbers a, b expressed as products of primes. Each prime factor of a occurs twice in the prime factorization of a^2. Also each prime factor of b occurs twice in the prime factorization of b^2.

This shows that while the prime number 2 either does not occur or occurs an even number of times on the left, it occurs an odd number of times, on the right. Thus, we arrive at a false statement.

It follows that the point P is such that no rational number corresponds to the same.

Measurements of Lengths Along a Line

While the set of natural numbers caters to the need for *counting* objects in any collection, the set of rational numbers *contributes* to the need for measuring entities like length, time, etc. It is, however, important to remark that the set of rational numbers does not prove adequate for measuring all lengths.

We need to extend the system of rational numbers to what is known as the system of *Real Numbers* to be able to measure all lengths. The system of rational numbers, as will be seen, is a sub-set of the set of real numbers.

The set of real numbers will be developed and studied in Algebra II.

40. SUMMARY.

$$Q = \{x, -x, 0 : x \in F\}$$

$$F = \left\{\frac{a}{b} : a \in N,\ b \in N\right\}$$

$$I = \{n, -n, 0 : n \in N\}$$

$$N \subset F \subset Q$$

$$N \subset I \subset Q.$$

Q is the set of rational numbers, F the set of fractions ahd I the set of integers.

F is a proper sub-set of Q and is the same as the set of positive members of Q.

N is a proper sub-set of I and is the same as the set of positive members of I.

Q_0 is the set of all non-zero rational numbers.

Addition Composition in Q

To each pair x, y of rational numbers, there corresponds a rational number denoted by $x + y$ called their sum This manner of associating to each pair x, y of rational numbers the rational number $x + y$ is called the Addition Composition in Q which has the following four properties :

1. *Addition composition is commutative, i.e.,*

$$x + y = y + x \ \forall\ x, y \in Q.$$

2. *Addition composition is associative, i.e ,*

$$(x + y) + z = x + (y + z),\ \forall\ x, y, z \in Q.$$

3. *Addition composition possesses the neutral element, 0, so that*

$$x + 0 = x \ \forall\ x \in Q.$$

4. To each rational number x corresponds a rational number denoted by x , and called the opposite or the negative of x such that

$$x + (-x) = 0,\ x \in Q.$$

Multiplication Composition in Q

To each pair x, y of rational numbers there corresponds a rational number denoted by xy and called their product. This manner of associating to each pair x , y of rational numbers the rational number xy is called the Multiplication Composition in Q having the following four properties :

5. *Multiplication composition is commutative, i.e.,*

$$xy = yx \ \forall \ x , y \in Q.$$

6. *Multiplication composition is associative, i.e.,*

$$(xy)\, z = x\, (yz) \ \forall \ x , y , z \in Q.$$

7. *Multiplication composition possesses the neutral element* 1 *so that*

$$x \times 1 = x \ \forall \ x \in Q.$$

8. *To each* **non-zero** *rational number x, there corresponds a rational number denoted by* $\frac{1}{x}$ *and called the reciprocal of x such that*

$$x \times \frac{1}{x} = 1 , \quad x \in Q_0.$$

Addition and multiplication jointly have property, *viz.,*

9. *Multiplication distributes addition, i.e.,*

$$x\,(y + z) = xy + xz \qquad \forall \ x , y , z \in Q.$$

Field of Rational Numbers

The set Q of rational numbers provided with the two compositions, addition and multiplication satisfying the above nine properties, is called the

FIELD OF RATIONAL NUMBERS.

Besides the field of rational numbers, there are two other very important fields of numbers, *viz.,*

(*i*) the field of real numbers, and

(*ii*) the field of complex numbers.

The study of these two fields will be taken up in Algebra II.

It is important to notice that the sets

N, F, I

of natural numbers, fractions and integers even though they possess the two compositions of addition and multiplication are not fields inasmuch as none of these satisfies all the nine field properties. It should be of interest to see as to which of these *nine* properties are not possessed by these three sets **N, F, I.**

It may not be difficult to see that

(*i*) N does not satisfy the properties 3 , 4 , 8.

(*ii*) **F** does not satisfy the properties 3 , 4.

(*iii*) I does not satisfy the property 8.

Subtraction and Division

The subtraction and division compositions in **Q** are defined as follows :

$$x - y = x + (-y) \qquad \forall\ x, y \in \mathbf{Q}$$

$$x - y = x \times \left(\frac{1}{y}\right) \qquad \forall\ x, y \in \mathbf{Q}, y \neq 0.$$

Note. The reader may try to understand the failure of one or both of subtraction and division in **N** , **F** , **I**

The Order Relation 'Is greater than' in Q

Besides the two compositions of addition and multiplication in **Q**, we also have a relation 'Is greater than' denoted by the symbol $>$. The relation satisfies the following properties .

10 *The relation satisfies the Trichotomy Law, i e., given any two rationals x, y we have one and only one of the following three possibilities.*

(*i*) $x > y$ (*ii*) $y > x$ (*iii*) $x = y$

11. *The relation is transitive, i e.,*

$$x > y \text{ and } y > z \Rightarrow x > z, \quad x, y, z \in \mathbf{Q}$$

Addition composition and the relation 'Is greater than' jointly satisfy the following property.

12. $$x > y \Leftrightarrow x + z > y + z, \quad x, y, z \in \mathbf{Q}.$$

Multiplication composition and the relation 'Is greater than' jointly satisfy the following property.

13. $$x > y \text{ and } z > 0 \Leftrightarrow xz > yz, \ x, y, z \in \mathbf{Q}, z > 0.$$

Ordered Field of Rational Numbers

The field of rational numbers is called an *Ordered Field* because of the set of rational numbers also possessing the four properties *vis-a-vis* the order relation 'Is greater than', besides the nine field properties.

It will be seen later on that the set of real numbers is also an ordered field.

While each of the two ordered fields of rational and real numbers possess the thirteen properties referred to above, there are properties which distinguish the ordered field of rational numbers from that of the ordered field of real numbers.

The set of complex numbers is not an ordered field.

Positive and Negative Rational Numbers

Definition $x > 0 \Leftrightarrow x$ is positive.

$x < 0 \Leftrightarrow x$ is negative.

Absolute value of a rational number x is denoted by $|x|$. Thus

$$|x| = \begin{cases} x & \text{if } x \text{ is positive} \\ -x & \text{if } x \text{ is negative} \\ 0 & \text{if } x \text{ is zero.} \end{cases}$$

41. SOME SPECIAL PRODUCTS.

In the following we give three special products which prove to be of importance in the study of second degree equations.

It will be seen that, as in other cases, we make use of the various basic properties of addition and multiplication compositions in the set Q of rational numbers in the following.

We show that $\forall\ x, y \in Q$.

(*i*) $$(x + y)^2 = x^2 + 2xy + y^2$$

(*ii*) $$(x - y)^2 = x^2 - 2xy + y^2$$

(*iii*) $$(x + y)(x - y) = x^2 - y^2.$$

Proof.

(*i*) We have

$$\begin{aligned}(x + y)(x + y) &= (x + y)\,x + (x + y)\,y \\ &= x\,(x + y) + y\,(x + y) \\ &= (xx + xy) + (yx + yy) \\ &= (x^2 + xy) + (xy + y^2) \\ &= x^2 + [xy + (xy + y^2)] \\ &= x^2 + [(xy + xy) + y^2] \\ &= x^2 + [1\cdot(xy) + 1\cdot(xy) + y^2] \\ &= x^2 + [(1 + 1)\,xy + y^2] \\ &= x^2 + [2xy + y^2] \\ &= x^2 + 2xy + y^2.\end{aligned}$$

The reader is advised to give a justification for each step in terms of the basic properties of Q. While we have tried in the above to put down every step and to employ only one basic property at each step, the reader should with some practice, be able to skip over some steps and do a good part of the process mentally.

(*ii*) We have $\forall\ x, y \in Q$,

$$\begin{aligned}(x - y)^2 &= (x - y)(x - y) \\ &= (x - y)\,x - (x - y)\,y \\ &= (x^2 - yx) - (xy - y^2) \\ &= x^2 - xy - xy + y^2 \\ &= x^2 - 2xy + y^2.\end{aligned}$$

Another method

$$\begin{aligned}[x + (-y)]^2 &= x^2 + 2x(-y) + (-y)^2 \\ &= x^2 - 2xy + y^2\end{aligned}$$

(*iii*) We have

$$\begin{aligned}(x+y)(x-y) &= (x+y)\,x - (x+y)\,y \\ &= (x^2+yx) - (xy+y^2) \\ &= x^2 + yx - xy - y^2 \\ &= x^2 - y^2.\end{aligned}$$

EXERCISES

1. Prove the following, the letters denoting arbitrary rational numbers.

(*i*) $(x+3y)^2 = x^2 + 6xy + 9y^2$

(*ii*) $(7x-5y)^2 = 49x^2 - 70xy + 25y^2$

(*iii*) $\left\{\frac{3}{2}x - \frac{7}{5}y\right\}^2 = \frac{9}{4}x^2 - \frac{21}{5}xy + \frac{49}{25}y^2$

(*iv*) $(\cdot 5x - 1\cdot 3y)^2 = \cdot 25x^2 - 1\cdot 3xy + 1\cdot 69y^2$

(*v*) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(*vi*) $(a+b-c)^2 = d^2 + b^2 + c^2 + 2ab - 2ac - 2bc$

(*vii*) $(a-b-c)^2 = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$

(*viii*) $(2x-3y)(2x+3y) = 4x^2 - 9y^2$

(*ix*) $(a^2b - ab^2)(a^2b + ab^2) = a^2b^2(a^2 - b^2)$

(*x*) $(x-y+3)(x+y-3) = x^2 - y^2 + 6y - 9$

(*xi*) $(a-3b+4c)(a+3b+4c) = a^2 - 9b^2 + 16c^2 + 8ac$

(*xii*) $(2x+y-z)(2x+y+z) = 4x^2 + y^2 - z^2 + 4xy$

(*xiii*) $(3a+7b-\frac{1}{2}c)(3a-7b-\frac{1}{2}c) = 9a^2 - 49b^2 + \frac{1}{4}c^2 - 3ac.$

2. For non-zero rational numbers x and y, show that

(*i*) $\left[x + \frac{1}{x}\right]^2 = x^2 + \frac{1}{x^2} + 2$

(*ii*) $\left[x - \frac{1}{x}\right]^2 = x^2 + \frac{1}{x^2} - 2$

(*iii*) $\left[x + \frac{1}{x}\right]\left[x - \frac{1}{x}\right] = x^2 - \frac{1}{x^2}$

(*iv*) $\left[x + \frac{1}{x}\right]\left[y + \frac{1}{y}\right] = xy + \frac{1}{xy} + \frac{x}{y} + \frac{y}{x}$

(*v*) $\left[x + \frac{1}{x}\right]\left[y - \frac{1}{y}\right] = xy - \frac{1}{xy} + \left[\frac{y}{x} - \frac{x}{y}\right]$

(*vi*) $\left[x - \frac{1}{x}\right]\left[y - \frac{1}{y}\right] = \left[xy + \frac{1}{xy}\right] - \left[\frac{x}{y} + \frac{y}{x}\right].$

3. Simplify the following. Also specify the values of x, y, z, a, b, c for which the expressions are not meaningful.

(i) $(-7a^2b)(3cba^2)$

(ii) $(-7x^3zy)\left[-\frac{1}{4}xyz^2\right]$

(iii) $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$

(iv) $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$

(v) $\dfrac{2x+3y}{\frac{2}{3y}+\frac{1}{x}}$

(vi) $\dfrac{(b+c)^2}{6bx}\cdot\dfrac{2bx}{(b+c)^2}$

(vii) $\dfrac{16x^2y^2}{3az^3}\cdot\dfrac{25z^2}{32xy^3}\cdot\dfrac{9xy}{5z}$

(viii) $\dfrac{(y+2x)^2}{ay-cy}\div\dfrac{y^2+2xy}{y^2a-y^2c}$

(ix) $\left[\dfrac{64a^2-b^2}{x^2-4}\cdot\dfrac{(x-2)^2}{16a+2b}\right]\div\dfrac{x^2-4}{(x+2)^2}$

(x) $\dfrac{4x^2+10}{x-3}\div\dfrac{6x^2+15}{x^2-9}$

(xi) $\dfrac{(x-y)^2}{x^2-y^2}$

(xii) $\dfrac{a^3-ab^2}{ab(a-b)^3}$

(xiii) $\dfrac{a^2-2ab+b^2}{a^2-b^2}$

(xiv) $\dfrac{y}{y-2}+\dfrac{2}{y+2}$

(xv) $\dfrac{1}{3x+4}+\dfrac{1}{3x-4}$

(xvi) $\dfrac{1}{7y-5}-\dfrac{1}{7y+5}$

(xvii) $\dfrac{a+b}{3ab}-\dfrac{2a+3}{6a^2}$

(xviii) $\dfrac{x+1}{x-1}+\dfrac{1-3x^2}{1-x^2}$

(xix) $\dfrac{x+3y}{x+2y}-\dfrac{x+2y}{x+3y}$

(xx) $\dfrac{x+2}{x+3}+\dfrac{x+3}{x+2}$.

Examples

1. Solve

$$5x-3(x-2)=3x-2(x-1),\ x\in\mathbf{Q}.$$

We have

$$\begin{aligned}
& 5x-3(x-2)=3x-2(x-1)\\
\Leftrightarrow\quad & 5x-3x+6=3x-2x+2\\
\Leftrightarrow\quad & 2x+6=x+2\\
\Leftrightarrow\quad & 2x+6-6=x+2-6\\
\Leftrightarrow\quad & 2x=x-4\\
\Leftrightarrow\quad & 2x-x=x-4-x\\
\Leftrightarrow\quad & x=-4.
\end{aligned}$$

The required truth set, thus, is

$$\{-4\}.$$

2. Solve

$$\frac{7x-1}{4} - \frac{1}{2}\left[2x - \frac{1-x}{2}\right] = 6\frac{1}{3}. \qquad x \in \mathbf{Q}.$$

We have

$$\frac{7x-1}{4} - \frac{1}{2}\left[2x - \frac{1-x}{2}\right] = \frac{19}{3}$$

$$\Leftrightarrow \quad \frac{7x-1}{4} - x + \frac{1-x}{4} = \frac{19}{3}$$

$$\Leftrightarrow 12\left[\frac{7x-1}{4} - x + \frac{1-x}{4}\right] = 12 \times \frac{19}{3}$$

$$\Leftrightarrow 3(7x-1) - 12x + 3(1-x) = 76$$

$$\Leftrightarrow \quad 21x - 3 - 12x + 3 - 3x = 76$$

$$\Leftrightarrow \quad (21 - 12 - 3)x - 3 + 3 = 76$$

$$\Leftrightarrow \quad 6x = 76$$

$$\Leftrightarrow \quad x = \frac{76}{6} = \frac{38}{3}.$$

It follows that $\frac{38}{3}$ is the solution of the given equation.

3. Find the truth set of

$$\frac{12x+1}{3} + (1+2x) > \frac{15x+4}{3} + x. \qquad x \in \mathbf{Q}.$$

We have

$$\frac{12x+1}{3} + (1+2x) > \frac{15x+4}{3} + x$$

$$\Leftrightarrow 3\left[\frac{12x+1}{3} + (1+2x)\right] > 3\left[\frac{15x+4}{3} + x\right]$$

$$\Leftrightarrow \quad 12x + 1 + 3(1+2x) > 15x + 4 + 3x$$

$$\Leftrightarrow \quad 12x + 1 + 3 + 6x > 18x + 4$$

$$\Leftrightarrow \quad 18x + 4 > 18x + 4$$

$$\Leftrightarrow \quad 18x > 18x$$

$$\Leftrightarrow \quad x > x.$$

But $x > x$ is false, as there exists no rational number x which is greater than itself.

The truth set is thus empty, *i.e.*, the truth set is ϕ.

4. Find the truth set of

$$\frac{4x-9}{5}+\frac{6x-3}{7}-\frac{10x+3}{2}>0,\ x\in Q.$$

We have

$$\frac{4x-9}{5}+\frac{6x-3}{7}-\frac{10x+3}{2}>0$$

$$\Leftrightarrow\quad 70\left[\frac{4x-9}{5}+\frac{6x-3}{7}-\frac{10x+3}{2}\right]>0.70=0$$

$$\Leftrightarrow\quad 14\,(4x-9)+10\,(6x-3)-35\,(10x+3)>0$$

$$\Leftrightarrow\quad 56x-126+60x-30-350x-105>0$$

$$\Leftrightarrow\quad (116-350)\,x-(126+30+105)>0$$

$$\Leftrightarrow\quad -234x-261>0$$

$$\Leftrightarrow\quad -234x-261+261>261$$

$$\Leftrightarrow\quad -234x>261$$

$$\Leftrightarrow\quad \left[-\frac{1}{234}\right](-234x)<\left(-\frac{1}{234}\right)(261)$$

$$\Leftrightarrow\quad x<-\frac{261}{234}.$$

The truth set, therefore, is

$$\left\{x : x<-\frac{261}{234},\ x\in Q\right\}.$$

EXERCISES

1. Find the truth sets of the following, given that $x \in Q$.

(i) $4x+5=3x-9$

(ii) $27x+41=29x-34$

(iii) $\frac{2x}{3}+\frac{3}{7}x-\frac{x}{5}=6$

(iv) $2{\cdot}3x+\frac{3}{7}={\cdot}7x-\frac{3}{16}$

(v) $\frac{x+10}{2}+\frac{15-5x}{3}=\frac{3\,(x+2)}{5}$

(vi) $\frac{2-3x}{3}+\frac{1+5x}{5}=\frac{3-8x}{4}$

(vii) $\frac{x+4}{7}=\frac{12x}{11}-(3x-5)$

(viii) $\frac{5x}{2}-\frac{7}{11}=\frac{7}{11}-\frac{4x}{3}$

(ix) $\frac{3x}{2} + \frac{8 - 4x}{7} = 3$

(x) $\frac{3}{4}(2x - 5) - \frac{5}{8}(3x + 1) = 1$

(xi) $\frac{3x + 5}{2} = 4x + 2 - \frac{9x - 4}{6}$

(xii) $1{\cdot}5(x - 5) - {\cdot}2(4x - 3) + 9 = 0$

(xiii) $\frac{1 - 6x}{10} - \frac{2x + 3}{6} - \frac{13 + 6x}{4} = 0$

(xiv) $\frac{2(x - 3)}{7} - \frac{2 - x}{3} = \frac{9x - 6}{63}$

(xv) $\frac{4x - 1}{6} - \frac{2x + 3}{9} = \frac{8x - 9}{18}$.

2. Find the truth sets of the following, given that $x \in Q$.

(i) $8x + 25 > 7x + 13$

(ii) $13x + 16 < 7x + 4$

(iii) $\frac{3}{4}x - \frac{4}{13} > \frac{5}{6}x + \frac{2}{11}$

(iv) ${\cdot}3x - {\cdot}75 > 1{\cdot}25 - {\cdot}7x$

(v) $\frac{3x}{5} - \frac{7x}{10} + \frac{3x}{4} \geqslant \frac{7x}{8} - 15$

(vi) $\frac{3x - 2}{3} - \frac{8x - 3}{4} \geqslant \frac{5x - 1}{5}$

(vii) $\frac{4x + 7 - (x - 6)}{3} + \frac{5x - 3}{3} \geqslant 0$

(viii) $\frac{x + 3}{2} + \frac{8 - 4x}{7} \leqslant 0$

(ix) $\frac{2x + 5}{4} - 2x \leqslant \frac{10x + 13}{8} + 1$

(x) $\frac{x - 2}{4} + \frac{5}{6} \leqslant x - \frac{2x - 1}{3} + \frac{1}{2}$.

3. Solve the following, assuming that x is a rational number other than those which render the statements meaningless.

(i) $\frac{2}{3x} - \frac{1}{x} = \frac{5}{9}$

(ii) $\frac{3}{x + 5} = \frac{1}{x - 5}$

(iii) $\frac{14}{x - 3} = \frac{12}{x + 4}$

(iv) $\frac{3}{4x} - \frac{2}{x} = \frac{4}{1 - 3x}$

(*v*) $\frac{4}{x-3} + \frac{3}{x+4} = 0.$

Examples

1. List the following sets.

(*i*) $\{x : |x| = 1, x \in Q\}$

(*ii*) $\{x : |2x - 1| = 5, x \in Q\}$.

Solution. (*i*) There are two rational numbers 1 and -1 whose absolute value is 1 and there is no other rational number whose absolute value is 1. It follows, therefore, that

$$\{x : |x| = 1, x \in Q\} = \{1, -1\}.$$

(*ii*) There are two rational numbers 5 and -5 whose absolute value is 5 and there is no other rational number whose absolute value is 5. It follows, therefore, that
$$|2x - 1| = 5,$$
if and only if

$$2x - 1 = 5 \text{ or } 2x - 1 = -5.$$

Now

$$2x - 1 = 5 \Leftrightarrow x = 3$$

and
$$2x - 1 = -5 \Leftrightarrow x = -2$$

$\therefore \quad \{x : |2x - 1| = 5, x \in Q\} = \{3, -2\}.$

2. List the following sets.

(*i*) $\{x : |x^2 - 5| = 4, x \in Q\}$

(*ii*) $\{x : |x^2 - 1| = 1, x \in Q\}$

Solution. (*i*) As in (*ii*) of example above, we have

$$|x^2 - 5| = 4 \text{ if and only if}$$
$$x^2 - 5 = 4 \text{ or } x^2 - 5 = -4.$$

Now
$$x^2 - 5 = 4 \Leftrightarrow x^2 = 9$$
$$\Leftrightarrow x = 3 \text{ or } x = -3$$

and
$$x^2 - 5 = -4 \Leftrightarrow x^2 = 1$$
$$\Leftrightarrow x = 1 \text{ or } x = -1.$$

Thus, we have

$$\{x : |x^2 - 5| = 4, x \in Q\} = \{3, -3, 1, -1\}.$$

The reader may verify that all the members of the set

$$\{3, -3, 1, -1\}$$

satisfy
$$|x^2 - 5| = 4.$$

(*ii*) Again $|x^2 - 1| = 1$, if and only if

$$x^2 - 1 = 1 \quad \text{or} \quad x^2 - 1 = -1.$$

Now
$$x^2 - 1 = 1 \Leftrightarrow x^2 = 2$$

and the solution set of $x^2 = 2$ in Q is empty.

Again $$x^2 - 1 = -1 \Leftrightarrow x^2 = 0$$
and the solution set of $x^2 = 0$ in Q is $\{0\}$.

Thus, we have $\{x : |x^2 - 1| = 1, x \in Q\} = \{0\}$.

3. Describe the following sets

(*i*) $\{x : |x| < 1, x \in Q\}$

(*ii*) $\{x : |2x + 3| < 2, x \in Q\}$

(*iii*) $\{x : |2x + 3| > 2, x \in Q\}$.

Solution. (*i*) We have,

if x is + ive or zero, $|x| = x$

and if x is − ive, $|x| = -x$.

Suppose now that x is + ive or zero. Then
$$|x| < 1 \text{ is equivalent to } x < 1. \qquad \ldots(a)$$
Again, when x is − ive, then
$$|x| < 1 \text{ is equivalent to } -x < 1$$
which is again equivalent to $x > -1$. .. (b)

Combining (a) and (b), we have
$$|x| < 1 \Leftrightarrow -1 < x < 1.$$

(*ii*) Just as in the case of (*i*) we have
$$|2x + 3| < 2 \Leftrightarrow -2 < 2x + 3 < 2.$$

Now
$$-2 < 2x + 3 \Leftrightarrow -\frac{5}{2} < x \qquad \ldots(c)$$
and
$$2x + 3 < 2 \Leftrightarrow x < -\frac{1}{2}. \qquad \ldots(d)$$

Combining (c) and (d), we have
$$|2x + 3| < 2 \Leftrightarrow -\frac{5}{2} < x < -\frac{1}{2}.$$

Thus,
$$\{x : |2x + 3| < 2, x \in Q\} = \{x : -\frac{5}{2} < x < -\frac{1}{2}, x \in Q\}.$$

(*iii*) If $(2x + 3)$ is non-negative, then
$$|2x + 3| = 2x + 3$$
and so *in this case*
$$|2x + 3| > 2 \Leftrightarrow 2x + 3 > 2$$
$$\Leftrightarrow x > -\frac{1}{2}.$$

Again *if* $2x + 3$ *is negative,* then

$$|2x+3| = -(2x+3)$$

so that

$$|2x+3| > 2 \Leftrightarrow -(2x+3) > 2$$
$$\Leftrightarrow 2x < -5$$
$$\Leftrightarrow x < -\frac{5}{2}.$$

Thus,

$$\{x : |2x+3| > 2, x \in Q\} = \left\{x : x > -\frac{1}{2}, x \in Q\right\}$$
$$\cup \left\{x : x > -\frac{5}{2}, x \in Q\right\}.$$

EXERCISES

1. List the following sets.

(*i*) $\{x : |x| = 2, x \in Q\}$ (*ii*) $\left\{x : |x| = \frac{5}{7}, x \in Q\right\}$

(*iii*) $\{x : |x-3| = 5, x \in Q\}$ (*iv*) $\{x : |x-7| = 4, x \in Q\}$

(*v*) $\{x : |7x-4| = 1, x \in Q\}$

(*vi*) $\left\{x : \left|\frac{2}{3}x + 5\right| = \frac{3}{4}, x \in Q\right\}$

(*vii*) $\{x : |x+7| = 3, x \in Q\}$ (*viii*) $\{x : |\cdot 5x - 2\cdot 3| = 1\cdot 7, x \in Q\}$

(*ix*) $\{x : |x-5| = 0, x \in Q\}$ (*x*) $\{x : |2x+5| = -3, x \in Q\}$.

2. List the following sets.

(*i*) $\{x : |x^2 - 3| = 13, x \in Q\}$ (*ii*) $\{x : |x^2 - 7| = 7, x \in Q\}$

(*iii*) $\{x : |x^2 - 8| = 8, x \in Q\}$ (*iv*) $\{x : |x^2 - 4| = 2, x \in Q\}$.

3. Describe the following sets.

(*i*) $\{x : |x| < 5, x \in Q\}$ (*ii*) $\{x : |3x+4| < 5, x \in Q\}$

(*iii*) $\{x : |7x-8| < 17, x \in Q\}$ (*iv*) $\{x : |x| > 2, x \in Q\}$

(*v*) $\{x : |x-7| < 3, x \in Q\}$ (*vi*) $\{x : |x-9| > 4, x \in Q\}$

(*vii*) $\{x : |3x-3| > 12, x \in Q\}$ (*viii*) $\left\{x : \left|\frac{1}{2}x + \frac{3}{5}\right| < \frac{1}{7}, x \in Q\right\}$

(*ix*) $\{x : |5x+4| < 0, x \in Q\}$ (*x*) $\{x : |2x-5| > 0, x \in Q\}$.

REVIEW EXERCISES

1. Given $AB = \left\{-\frac{5}{7}, -\cdot 7, 1.4, -\cdot 3\cdot 75, 0, -\frac{21}{6}, 2\cdot 4\right\}$

and

$$B = \left\{\frac{2}{3}, \frac{7}{11}, -\frac{13}{4}, -\frac{2}{3}, -\cdot 7, -3\cdot 7 - \frac{4}{15}, 2\cdot 16\right\}$$

find the greatest and the least members of the sets

$$A, B, A \cup B, A \cap B.$$

2. Put down the greatest and the least members, if any, of the following sets.

$$A = \{x : -5 \leqslant x < -3, x \in \mathbf{Q}\}$$
$$B = \{x : -5 < x \leqslant -3, x \in \mathbf{Q}\}$$
$$C = \{x : -5 \leqslant x \leqslant -3, x \in \mathbf{Q}\}$$
$$D = \{x : -5 < x < -3, x \in \mathbf{Q}\}$$
$$E = \{x : -1 < x \leqslant 0, x \in \mathbf{Q}\}$$
$$F = \{x : -1 < x < 0, x \in \mathbf{Q}\}$$
$$G = \{x : -1 \leqslant x \leqslant 0, x \in \mathbf{Q}\}$$
$$H = \{x : -1 \leqslant x < 0, x \in \mathbf{Q}\}$$

3. Show that $x > y \Rightarrow -7 - 5y > -7 - 5x \;\forall\; x, y \in \mathbf{Q}_0$.

4. Show that $x > y \Rightarrow \dfrac{1}{y} > \dfrac{1}{x} \;\forall\; +\text{ive } x, y \in \mathbf{Q}$

5. Prove that $x > y \Rightarrow x^2 > y^2 \;\forall\; +\text{ive } x, y \in \mathbf{Q}$.

6. Prove that $x > y \Rightarrow x^2 < y^2 \;\forall\; -\text{ive } x, y \in \mathbf{Q}$

7. Show that $x^2 - 1 > 0 \;\forall\; x > 1$ and $\forall\; x < -1, x \in \mathbf{Q}$.

8 Show that $x^2 - 1 < 0 \;\forall\; -1 < x < 1, x \in \mathbf{Q}$.

9. Given that $x, y, a, b \in \mathbf{Q}$ and the expressions are meaningful, simplify.

(i) $\dfrac{2}{4 - x^2} \cdot \dfrac{2 - x}{2}$ (ii) $\dfrac{2x + 2y}{5} \cdot \dfrac{15}{\frac{3}{4}x + \cdot 75y}$

(iii) $\dfrac{3a + 2b}{a - b} \cdot \dfrac{3a - 2b}{a + b}$ (iv) $\dfrac{4 - a^2}{7a - 14} \quad \dfrac{8}{a + 2}$

(v) $\dfrac{x^2 + xy}{y - x} \div \dfrac{x + y}{x^2 - xy}$.

10. Given $x \in \mathbf{Q}$, solve the following equations.

(i) $8x + \dfrac{3}{2} = 9x + \dfrac{5}{4}$ (ii) $\dfrac{6x}{13} + 9 = \dfrac{x}{12} - \dfrac{3}{4}$

(iii) $\dfrac{23x}{27} - \dfrac{13}{9} = \dfrac{13}{9} - \dfrac{4}{5}x$ (iv) $\dfrac{7x - 1}{7} + \dfrac{4x - 3}{4} = \dfrac{10x - 7}{5}$

(v) $\dfrac{x + 4}{9} = \dfrac{2x}{5} - (3x + 2)$

(vi) $\dfrac{3}{11}(3x + 8) = \dfrac{7}{8}(6x + 15)$

(vii) $\dfrac{10x + 7}{4} + \dfrac{9 - 2x}{5} + \dfrac{x - 8}{7} = 0$

(*viii*) $\dfrac{2}{x-7}+\dfrac{3}{x+7}=0$

(*ix*) $\dfrac{2}{3x+4}-\dfrac{5}{4x-7}=0$

(*x*) $\dfrac{(x-1)}{(x-2)}-\dfrac{x-3}{(x-4)}=0$

(*xi*) $\dfrac{x-a}{x-b}-\dfrac{x-c}{x-d}=0$

(*xii*) $\dfrac{x+a}{x+b}-\dfrac{x+c}{x+d}=0.$

Note. In (*viii*) — (*xii*), it is assumed that the algebraic expressions are meaningful.

11. Given $x \in \mathbf{Q}$, find the truth sets of the following.

(*i*) $17x - 15 > 19x - 15$

(*ii*) $\dfrac{5}{7}x-\dfrac{2}{3}>\dfrac{7}{9}-\dfrac{2}{21}x$

(*iii*) $\dfrac{9x+5}{5}-\dfrac{x-4}{3}\geqslant 0$

(*iv*) $\dfrac{3x+4}{5}-3x\leqslant\dfrac{8x+15}{6}+1$

(*v*) $|2x-9|=4$

(*vi*) $|3x+4|=7$

(*vii*) $|x^2-13|=12$

(*viii*) $|x^2-18|=18$

(*ix*) $|x^2-3|=4$

(*x*) $|x^2-4|=3$

(*xi*) $\left|5x-\dfrac{2}{3}\right|<4$

(*xii*) $\left|3x-\dfrac{2}{5}\right|>\dfrac{3}{4}$

(*xiii*) $\left|2x+\dfrac{3}{5}\right|<7$

(*xiv*) $\left|4x+\dfrac{3}{7}\right|>\dfrac{1}{2}$

(*xv*) $(x-2)(x-3)>0$

(*xvi*) $(x-2)(x-3)<0$

(*xvii*) $(x-2)(x-3)=0$

(*xviii*) $x(x-1)>0$

(*xix*) $x(x-1)<0$

(*xx*) $x(x-1)=0.$

12. Deduce the following results *directly* from the thirteen ordered field properties of the set **Q** of rational numbers and the definition of positive and negative rational numbers as given in the summary.

(*i*) The sum of two positive rational numbers is positive.

[**Hint.** $x>0$ and $y>0 \Rightarrow x+y>0+0=0$].

(*ii*) The sum of two negative rational numbers is negative.

(*iii*) $x>y \Leftrightarrow x-y$ is positive.

[**Hint.** $x>y \Leftrightarrow x+(-y)>y+(-y)$].

(*iv*) $x<y \Leftrightarrow x-y$ is negative.

(*v*) A rational number, x, is positive or negative according as its opposite, $-x$, is negative or positive.

[**Hint.** $x>0 \Leftrightarrow x+(-x)>0+(-x) \Leftrightarrow 0>0-x=-x$.]

(*iv*) $0.\,x=0 \ \forall\ x\in\mathbf{Q}$.

[**Hint.** $x(0+0)=x0+x0$

$\Rightarrow x0=x0+x0$

$\Rightarrow \quad x0+[-(x0)]=[x0+x0]+[-(x0)]$

(*vii*) $xy = 0 \Leftrightarrow x = 0$ or $y = 0$ or both x, y are 0.

(*viii*) $$\left.\begin{aligned} x(-y) &= -(xy) \\ (-x)(-y) &= xy \end{aligned}\right\} \forall\, x, y \in Q.$$

[Hint. Use the distributive law.]

(*ix*) The product of two positive and of two negative numbers is positive.

(*x*) The product of two numbers, one positive and the other negative, is negative.

(*xi*) $$\left.\begin{aligned} |x| &\geqslant x \\ |x| &\geqslant -x \end{aligned}\right\} \forall\, x \in Q.$$

(*xii*) $-|x| \leqslant x \leqslant |x| \;\forall\, x \in Q.$

(*xiii*) $|x|$ is the greater of the two numbers $x, -x$.

(*xiv*) $|x+y| \leqslant |x| + |y| \;\forall\, x, y \in Q.$

[Hint. We have

$$\begin{aligned} & -|x| \leqslant x \leqslant |x| \text{ and } -|y| \leqslant y \leqslant |y| \\ & \Rightarrow -(|x|+|y|) \leqslant x+y \leqslant |x|+|y|.] \end{aligned}$$

(*xv*) $|xy| = |x|\;|y| \;\forall\, x, y \in Q.$

(*xvi*) $|x-a| < b \Leftrightarrow a-b < x < a+b.$

(*xvii*) $-(x+y) = -x-y \;\forall\, x, y \in Q.$

(*xviii*) $\dfrac{1}{xy} = \dfrac{1}{x} \cdot \dfrac{1}{y} \;\forall\, x, y \in Q_0.$

5

Linear Equations
Single and Systems

42. INTRODUCTION

We have developed the system of rational numbers as an ordered field in *Chapter 4*. It is now proposed to study a single linear equation and systems of linear equations, in the context of the system of rational numbers. It may be remembered that the reader has already been concerned with linear equations in one variable in chapters 1, 3 and 4, and he will now notice the power of the system of rational numbers in respect of doing away with the inadequacies of the system of natural numbers and fractions in dealing with linear equations. While the general study is restricted to systems involving two variables, the case of the equations in three variables is taken up through specific examples.

The problem of consistency of linear equations is also discussed at length and through it the reader is acquainted with what is commonly known as *Elimination*.

In the context of open statements, equations or equalities, a *variable* is often referred to as an *unknown*.

43. LINEAR EQUATIONS OVER THE FIELD OF RATIONAL NUMBERS IN ONE VARIABLE.

An equation in one unknown x is said to be *linear* over the field of rational numbers if it is equivalent to an equation of the form

$$ax + b = 0$$

where a and b are rational numbers, $a \neq 0$. The number a is called the *coefficient* of x and b is called the *constant term* of the equation (1). Thus, for example,

$$3x + \frac{5}{2} = 0$$

is a linear equation in which the coefficient of x is the number 3 and the constant term is $\frac{5}{2}$. The equation (1) is called the standard form of a linear equation in x.

Examples. 1. Show that

$$3x + 7 = \frac{2}{5}x + \frac{3}{2}$$

is a linear equation.

We have

$$3x + 7 = \frac{2}{5}x + \frac{3}{2}$$

$$\Leftrightarrow 3x + 7 - \left[\frac{2}{5}x + \frac{3}{2}\right] = \left[\frac{2}{5}x + \frac{3}{2}\right] - \left[\frac{2}{5}x + \frac{3}{2}\right]$$

$$\Leftrightarrow \left[3 - \frac{2}{5}\right]x + \left[7 - \frac{3}{2}\right] = 0$$

$$\Leftrightarrow \quad \frac{13}{5}x + \frac{11}{2} = 0$$

Thus, the given equation, being equivalent to the equation,

$$\frac{13}{5}x + \frac{11}{2} = 0$$

is a linear equation.

2 Consider the equation

$$\frac{3}{x-2} + \frac{4}{3} = 0.$$

Surely, for $\frac{3}{x-2}$ to be meaningful the domain of the variable x must exclude the number 2, so that the domain of x consists of the set of rational numbers excluding the number 2.

We have

$$\frac{3}{x-2} + \frac{4}{3} = 0$$

$$\Leftrightarrow (x-2)\left\{\frac{3}{x-2} + \frac{4}{3}\right\} = 0, \qquad (x - 2 \text{ not being } 0)$$

$$\Leftrightarrow \quad 3 + \frac{4}{3}(x-2) = 0$$

$$\Leftrightarrow \quad 3 + \frac{4}{3}x - \frac{8}{3} = 0$$

$$\Leftrightarrow \frac{4}{3}x + \frac{1}{3} = 0.$$

Thus, the given equation is linear.

We repeat that an equation obtained from a given equation,

(*i*) by adding the same rational number to both sides,

(*ii*) by multiplying both sides by the same non-zero rational number, is equivalent to the given equation.

We note that main steps in the two examples above are respectively,

(*i*) adding $-\left[\frac{2}{5}x + \frac{3}{2}\right]$ to both sides

(*ii*) multiplying both sides by $(x - 2)$, which is assumed to be not zero.

Note. It may be noted that subtracting the same rational number from the two sides essentially means adding its opposite to the two sides and dividing both sides of an equation by the same non-zero rational number is the same as multiplying the two sides by its reciprocal. Thus, instead of talking of four basic principles of obtaining equivalent equations, as was done on page 35 of chapter 1, we are now in a position to talk in terms of addition and multiplication only.

EXERCISES

1. Reduce the following equations to the standard linear form.

(*i*) $3x = 2$ (*ii*) $4 = \frac{2}{3}x$

(*iii*) $ax = b$ (*iv*) $5x + 4 = 2$

(*v*) $3x + \frac{1}{2} = \frac{7}{4}x$ (*vi*) $x - \frac{2}{3} = {\cdot}5x - \frac{1}{4}$

(*vii*) $ax + b = c, (a \neq 0)$ (*viii*) $ax + b = cx; (a \neq c)$

(*ix*) $ax + b = cx + d, (a \neq c)$.

2. Show that the following equations are linear.

(*i*) $\frac{x}{5} + \frac{x-4}{5} = 8$ (*ii*) $\frac{x-1}{10} + \frac{x-2}{15} = \frac{x-3}{20}$

(*iii*) ${\cdot}15x + {\cdot}3 = {\cdot}75$ (*iv*) $\frac{7x-1}{4} + \frac{1}{3}\left[12x + \frac{1-x}{2}\right] = 0$

(*v*) $\frac{7-x}{7} - \frac{8-x}{8} + \frac{9-x}{9} = 1$

(*vi*) ${\cdot}09x - {\cdot}63 = \frac{{\cdot}17x - {\cdot}75}{4} + {\cdot}01x.$

(*vii*) $\frac{3+x}{3} + \frac{4+x}{4} = \frac{5+x}{5} + 2$

(*viii*) $(x+1)(x+2)=(x+3)(x+5)$

(*ix*) $(x-3)(x-4)=(x-1)(x-5)$

(*x*) $(2x+1)(8x-3)=(4x-2)^2$.

3. Show that the following equations are not linear.

(*i*) $(x+3)(2x+5)=(2x+1)(3x-1)$

(*ii*) $x^2-4=2x(x+5)$.

4. Which of the following equations are linear and which are not ? Specify in each case the domain of the variable x.

(*i*) $\dfrac{3}{x-1}+\dfrac{4}{x}=0$ (*ii*) $\dfrac{1}{x-a}+\dfrac{1}{x-b}=0$

(*iii*) $\dfrac{1}{x-1}+\dfrac{1}{x+3}=2$ (*iv*) $5-\dfrac{7}{2x+1}=0$

(*v*) $\dfrac{11}{x-7}=\dfrac{7}{x-4}$ (*vi*) $\dfrac{x-3}{2x+5}+\dfrac{1}{2}=0$

(*vii*) $\dfrac{x-2}{x+5}=\dfrac{x+3}{x-4}$ (*viii*) $\dfrac{x-2}{x+1}+3=\dfrac{x+4}{x-1}$.

Note. An equation which is equivalent to

$$ax^2+bx+c=0$$

where $a \neq 0$ and $a, b, c \in \mathbf{Q}$ is called a *quadratic* equation. A detailed study of such an equation will be taken up in the next chapter.

5. Which of the equations in Ex. 4 are quadratic equations ?

Solution of a Linear Equation in One Variable

Given a linear equation in one variable, it can always be put in the equivalent standard form

$$ax+b=0, \qquad a, b \in \mathbf{Q} \qquad a \neq 0.$$

Thus, in order to be able to solve any given linear equation, we should know the solution of this standard linear equation. We seek in the following a solution of this linear equation, where a and b are some given rational numbers, $a \neq 0$ and the domain of the variable is the set $\mathbf{Q}$.

The process consists in obtaining a chain of equivalent equations such that the truth-set of the last of these is obvious. In fact, we have

$$ax+b=0$$

$$\Leftrightarrow (ax+b)+(-b)=0+(-b)$$

(Adding $-b$ to both sides)

$$\Leftrightarrow \qquad ax=-b.$$

Also because $a \neq 0$, it admits of the reciprocal $1/a$.

Again, we have

$$ax = -b$$

$$\Leftrightarrow \frac{1}{a}(ax) = \frac{1}{a}(-b)$$

$\left(\text{Multiplying both sides by } \frac{1}{a}\right)$

$$\Leftrightarrow \qquad x = -\frac{b}{a}.$$

We obtain, therefore,

$$ax + b = 0 \Leftrightarrow x = -\frac{b}{a}.$$

Now the solution set of the equation

$$x = -\frac{b}{a}$$

is the set consisting of the member, $-\frac{b}{a}$, so that we see that the solution set of the given equation

$$ax + b = 0$$

is

$$\left\{-\frac{b}{a}\right\}$$

and as such the truth set of the given equation is one elementic. Thus, a linear equation in one variable has a unique solution in the set Q of rational numbers.

Note. It is very important to note that we could not have the step of adding $(-b)$ to both sides unless we were dealing with rational numbers (or at least integers) and that we could not have the step of multiplying both sides by $1/a$ unless we were dealing with rational numbers (or at least fractions). In order, however, to be able to perform both these steps, it is essential that the domain of the variable is is the set Q of rational numbers.

EXERCISES

1. Solve the equations of Ex. 1 on page 200.
2. Solve the equations of Ex. 2 on page 200.
3. Find the solution set of Ex. 3 on page 201.
5. Find the truth set of those Equtions of Ex. 4 on page 201 which are linear.

Consistency of Two Linear Equation in One Variable

Consider two linear equations

$$ax + b = 0 \qquad a, b \in Q, a \neq 0$$
$$cx + d = 0 \qquad c, d \in Q, c \neq 0,$$

the domain of the variable x being the set Q. We say that the equations are consistent if there exists a value of x which satisfies both the equations. In other words, the two equations are consistent if the intersection of the solution sets of the two equations is non-empty.

Now, the truth set of the first equation is

$$\left\{-\frac{b}{a}\right\}$$

and that of the second equation is

$$\left\{-\frac{d}{c}\right\}.$$

The intersection of these two sets will be non-empty, if and only if

$$-\frac{b}{a}=-\frac{d}{c}.$$

But
$$-\frac{b}{a}=-\frac{d}{c}$$
$$\Leftrightarrow \quad bc = ad.$$

Thus, the two linear equations are consistent, if and only if

$$bc = ad. \tag{1}$$

We see that the two equations are consistent if and only if they are equivalent.

The process of finding the condition of consistency of the two given equations is called *Elimination* and (1) is called the *Eliminant* of the two equations inasmuch as (1) does not contain the variable x.

EXERCISES

1. Find the condition of consistency of the following pairs of equations, the domain of the variable x being the set Q.

(*i*) $x - a = 0, x - b = 0$ (*ii*) $x + a = 0, x + b = 0$
(*iii*) $x - l = 0, x + m = 0$ (*iv*) $x + l = 0, x - m = 0$
(*v*) $x - a = 0, bx = c$ (*vi*) $x - a = 0, bx + c = 0$
(*vii*) $x + a = 0, bx = c$ (*viii*) $x + a = 0, bx + c = 0$
(*ix*) $px + q = 0, lx = m$ (*x*) $px - q = 0, lx + m = 0$
(*xi*) $px - q = 0, lx - m = 0$ (*xii*) $ax + b = c, dx = e.$

2. Which of the following pairs of equations are consistent and which are not ?

(*i*) $x - 3 = 0, 7x = 21$ (*ii*) $3x + 2 = 0, 2x = -\frac{4}{3}$

(*iii*) $3x + 5 = 0, 8x + 21 = 0$ (*iv*) $5x = 4, \frac{5}{8}x - \frac{1}{2} = 0.$

3. Assuming a, b as different non-zero rational numbers, find which of the following pairs of equations are consistent.

(*i*) $2x - 3a = 0,\ \frac{1}{3}x = \frac{a}{2}$ (*ii*) $ax + 2b = 0,\ 3x + \frac{b}{a} = 0$

(*iii*) $ax + b = 0,\ x = \frac{b}{a}$ (*iv*) $ax + b = 0,\ a^2x = ab$

(*v*) $ax + b = 0, \frac{a^2}{b^2}x + \frac{a}{b} = 0$ (*vi*) $ax + b = 0,\ a^2x + b^2 = 0.$

4. Which of the following pairs of equations are consistent and which are not ?

(*i*) $\frac{x-3}{7} + \frac{5-x}{3} = 0,\quad \frac{x-3}{3} + \frac{5-x}{7} = 0$

(*ii*) $\frac{7-}{7} - \frac{8-x}{8} + \frac{9-x}{9} = 1,\ \frac{3-x}{3} - \frac{4-x}{4} + \frac{5-x}{5} = 1$

(*iii*) $\frac{2x+3}{3} + \frac{x+8}{8} - \frac{3x+11}{11} = 1,\ \frac{x+2}{2} - \frac{5x+7}{7} + \frac{4x+9}{9} = 1$

(*iv*) $\frac{3x-1}{4} + \frac{1}{7}\left[9x + \frac{1-x}{3}\right] = 0, \frac{4x-2}{5} + \frac{1}{8}\left[10x + \frac{1-x}{4}\right] = 0.$

44. LINEAR EQUATION IN TWO VARIABLES.

An equation in two variables x, y is said to be *linear* if it is equivalent to an equation of the form

$$ax + by + c = 0 \tag{1}$$

where a, b, c are rational numbers ; a and b being not both zero. a and b are called the co-efficients of x and y respectively and c is called the constant term of the equation (1). Also (1) is said to be the standard form of a linear equation in two variables x and y.

As an illustration, let us consider the equation,

$$(2x - 3) + (5 - 4y) = \frac{1}{2}y + 3x + 5.$$

We have the following chain of equivalent equations,

$$2x - 4y + 2 = 3x + \frac{1}{2}y + 5$$

$$\Leftrightarrow 2x - 4y + 2 - \left(3x + \frac{1}{2}y + 5\right) = 0$$

$$\Leftrightarrow -x - \frac{9}{2}y - 3 = 0$$

$$\Leftrightarrow x + \frac{9}{2}y + 3 = 0.$$

The last of these being of the form (1), we see that the given equation is linear.

For another illustration, let us consider the equation

$$\frac{7x-5}{3-11y}=4.$$

Certainly, the variable y cannot take the value $3/11$ inasmuch as this will render $(3-11y)$ zero. Having imposed this restriction on the domain for y, we have the following chain of equivalent equations,

$$\begin{aligned} & 7x-5=4\,(3-11y) \\ \Leftrightarrow\ & 7x-5=12-44y \\ \Leftrightarrow\ & 7x+44y-17=0 \\ \Leftrightarrow\ & 7x+44y+(-17)=0 \end{aligned}$$

the last of which is of the form (1). The given equation is, therefore, linear.

EXERCISES

1. Reduce to the standard linear form the following equations.

(*i*) $2x+3y=4$ (*ii*) $2x+4=3y$

(*iii*) $3y+4=2x$ (*iv*) $2x+2=3y+5$

(*v*) $x-y-3=2x+3y-5$ (*vi*) $3x-2y+5=7x+5$

(*vii*) $\dfrac{3x-5}{2}+\dfrac{3-4y}{4}=\dfrac{3y}{4}$

(*viii*) $\dfrac{x+2y}{3}+\dfrac{3y-4}{12}=\dfrac{7x+11y+3}{2}.$

2. Which of the following equations are linear and which are not? Also state the restrictions you have to impose on the domains of the variables x and y in each case.

(*i*) $\dfrac{x-3}{y}+4=0$ (*ii*) $\dfrac{2y+3}{x-5}+7=0$

(*iii*) $(x+4)+\dfrac{3}{y}=0$ (*iv*) $\dfrac{3}{2x+5}+\dfrac{7}{6-13y}=0$

(*v*) $\dfrac{x+7}{2y+3}-\dfrac{3x-5}{6y-1}=0$ (*vi*) $\dfrac{2x-3}{7y+11}+\dfrac{3+8x}{9-28y}=0.$

Solution of a Linear Equation in Two Variables

Consider the linear equation

$$2x-3y+4=0$$

where the domain of each of the two variables x, y is the set Q of rational numbers. This equation is equivalent to

$$\begin{aligned} & 2x+4=3y \\ \Leftrightarrow\ & \frac{2x+4}{3}=y. \end{aligned}$$

By giving to x any value as a member of Q, we obtain a corresponding value of y also in Q, as for example, if x takes the value 0, the value of y is $\frac{4}{3}$. Then we say that the **ordered pair**

$$\left(0, \frac{4}{3}\right)$$

is a solution of the given equation. Similarly, we may see that

$$(1, 2), \left(2, \frac{8}{3}\right), \left(-1, \frac{2}{3}\right), (-2, 0)$$

are some other ordered pairs of rational numbers which are solutions of the equation.

It is not at all difficult to verify that the ordered pair

$$(2, 1)$$

is not a solution of the given equation.

Caution. The reader must carefully note that whereas (1, 2) is a solution of the equation, (2, 1) is not. Although the two numbers involved in both these ordered pairs are the same, yet the two are different. Thus, we have a distinction between a pair and an ordered pair of numbers. In an ordered pair, the order of occurrence of the two numbers is very important, *i.e.*, the first or the left member and the second or the right member is specified in an ordered pair. Such a specification is not there when we describe just a pair of numbers.

We have already listed five ordered pairs of rational numbers which are solutions of the given equation. At this stage a very natural question arises.

'Can we write down all such ordered pairs which are solutions of the given equation ?' The answer is an emphatic 'No' inasmuch as any value could be given to x and then we can find the corresponding value of y. Thus, the truth set of the given equation, consisting of ordered pairs of rational numbers, is infinite.

Before discussing the solution of the standard linear equation in two variables, we study in the following the notion of *Ordered Pairs* in greater detail.

The Set $\mathbf{Q} \times \mathbf{Q}$ *or* $\mathbf{Q}^2$

Definition. *The set* $\mathbf{Q} \times \mathbf{Q}$ *or* $\mathbf{Q}^2$ *is defined as the set of all ordered pairs* (a, b) *where* $a, b \in \mathbf{Q}$. *The number a is called the first member or the left member and b is called the second member or the right member of the ordered pair* (a, b).

For example,

$$(1, 1), (0, 1), \left(\frac{1}{2}, -\frac{5}{2}\right), (-\cdot 5, -1\cdot 75)$$

are some members of the set $\mathbf{Q} \times \mathbf{Q}$.

In the set builder notation. $Q \times Q$ can be symbolically described in the following manner

$$Q \times Q = \{(a\,,b) \cdot a \in Q\,,\, b \in Q\}.$$

Definition. *Two ordered pairs are said to be equal, the same or identical if and only if their first members are equal and the second members are equal.*

Thus, for example, the ordered pairs

$$(3\,,5)\,,\,(7-4\,,2+3)$$

are equal but

$$(3\,,5)\,,\,(5\,,3)$$

are not equal. We see that the two ordered pairs are different even when the numbers constituting the pairs are the same The order of occurrence of the members of the ordered pair is very important.

In general we have,

$$(a\,,b) = (c\,,d) \Leftrightarrow a = c \text{ and } b = d.$$

EXERCISES

1. Put down 10 different ordered pairs of rational numbers.
2. State whether the two ordered pairs are the same or not.

(*i*) $(1\,,1)\,,\left(\frac{1}{2}+\frac{1}{2}\,,2{\cdot}5-1{\cdot}5\right)$ (*ii*) $(3\,,4)\,,(3\,,7)$

(*ii*) $(7\,,2)\,,(5\,,2)$ (*iv*) $(11\,,13)\,,(13-2,11+2)$

(*v*) $(13\,,11)\,,(13-2\,,11+2)$ (*vi*) $(14\,,19)\,,\left(7\,,\frac{19}{2}\right)$

(*vii*) $(24\,,36)\,,(6\,,9)$ (*viii*) $(-4\,,8)\,,(4\,,-8)$

(*ix*) $(a\,,-b)\,,(-a\,,b)$ (*x*) $(a\,,b)\,,(-a\,,-b)$

(*xi*) $(a\,,b)\,,(a+b\,,a+b)$ (*xii*) $(a\,,b)\,,(a+c\,,b+c)$.

[In (*ix*) to (*xii*) it is assumed that $a\,,b\,,c$ are different non-zero rational numbers.]

Example

Put down the set of ordered pairs for which

$$\frac{22x+y-11}{4-7y}$$

is not meaningful

Solution. For the given expression to be meaningful, we must have $4-7y \neq 0$. Thus, the given expression will not be meaningful if $4-7y=0$ *i.e.*,

if $$y = \frac{4}{7}$$

The set of ordered pairs for which the given expression is not meaningful is

$$\left\{\left(x, \frac{4}{7}\right) : x \in Q\right\}.$$

This set is essentially a sub set of Q × Q. The ordered pairs for which the given expression is meaningful will be members of the set

$$\left\{(x, y) : x, y \in Q, y \neq \frac{4}{7}\right\}.$$

EXERCISES

1. Put down the sets of ordered pairs for which the following expressions are not meaningful.

(i) $\frac{3x-4}{y}$ (ii) $\frac{x}{2y+5}$

(iii) $\frac{x-y+3}{7y-11}$ (iv) $\frac{4-2y}{x}$

(v) $\frac{y}{3x+5}$ (vi) $\frac{2x+3y+7}{5-8x}$.

2. Put down the sets of ordered pairs for which the expressions of Ex. 1 are meaningful.

3. Put down at least five solutions of each of the following linear equations.

(i) $x + y + 1 = 0$ (ii) $x + y + 5 = 0$

(iii) $x - y + 3 = 0$ (iv) $-x + y - 4 = 0$

(v) $2x + 3y - 5 = 0$ (vi) $3x + 4y + 7 = 0$

(vii) $3x - 7y + 4 = 0$ (viii) $-4x + 5y - 9 = 0$

(ix) $\frac{1}{2}x - \frac{3}{4}y + 3 = 0$ (x) $\cdot 75x - 1\cdot 25y + 3\cdot 5 = 0$.

4. Find at least two solutions of each of the linear equations of Exercise 1 on page 205.

5. Find at least three solutions of those of the equations Exercise 2 on page 205, which are linear.

Solution of the Standard Linear Equation in Two Variables

The standard linear equation in two variables is

$$ax + by + c = 0,$$

where the domain of each of the variables x, y is Q and a, b, c are given rational numbers such that a and b are not both zero.

Let $a \neq 0$.

We have then,

$$\begin{aligned} & ax + by + c = 0 \\ \Leftrightarrow\ & ax + by + c - (by + c) = -(by + c) \\ \Leftrightarrow\ & \qquad\qquad\qquad ax = -(by + c). \end{aligned}$$

Also, because $a \neq 0$, $1 / a$ exists, so that

$$\begin{aligned} & ax = -(by + c) \\ \Leftrightarrow\ & \frac{1}{a}(ax) = \frac{1}{a}\{-(by + c)\} \\ \Leftrightarrow\ & x = -\frac{by + c}{a}. \end{aligned}$$

By giving to y any value as a member of Q we can obtain a value of x. Thus, for example, if a is the value of x corresponding to the value k of y, then

$$h = -\frac{bk + c}{a}.$$

The orderded pair (h, k) is therefore a solution of this given equation. By giving to y different values, we get different values of x unless $b = 0$. The truth set of the given equation is

$$\{(h, k) : h = -\frac{bk + c}{a}, h, k \in Q\}.$$

This truth set is certainly a sub-set of $Q \times Q$.

The reader may, similarly, see that if $b \neq 0$,

$$\begin{aligned} & ax + by + c = 0 \\ \Leftrightarrow\ & y = -\frac{ax + c}{b}. \end{aligned}$$

By giving to x any value, we can obtain a corresponding value of y. The ordered pair (u, v) such that

$$v = -\frac{au + c}{b}$$

is a solution of the given equation. The truth set of the equation is, therefore,

$$\{(u, v) . v = -\frac{au + c}{b}, u, v \in Q\}$$

Note If none of a and b is zero, any one of the two methods can be used because in such a case

$$\begin{aligned} & ax + by + c = 0 \\ \Leftrightarrow\ & x = -\frac{by + c}{a} && \ldots (i) \\ \Leftrightarrow\ & y = -\frac{ax + c}{b}. && \ldots (ii) \end{aligned}$$

Any ordered pair (x, y) obtained from (i) will satisfy (ii) and *vice versa*

45. SYSTEMS OF LINEAR EQUATIONS IN TWO VARIABLES

Consider two linear equations in x and y

$$ax + by + c = 0 \qquad \text{...(1)}$$

$$a'x + b'y + c' = 0 \qquad \text{...(2)}$$

where a, b, c ; a', b', c' are all rational numbers and the domain of each of the variables x and y is the set Q. The two open statements (1) and (2) can be *compounded* in two different ways to give rise to the following two different compoud statements.

$$ax + by + c = 0 \quad \text{or} \quad a'x + b'y + c' = 0 \qquad \text{...(3)}$$

$$ax + by + c = 0 \quad \text{and} \quad a'x + b'y + c' = 0 \qquad \text{...(4)}$$

Thus, the statement (3) is true whenever (1) is true or (2) is true and the statement (4) is true whenever (1) and (2) are *both* true. The truth set of (3) will consist of the union of the truth sets of (1) and (2) and the truth set of (4) will consist of the common solutions of (1) and (2), *i.e.*, the truth set (4) is the intersection of the truth sets of (1) and (2). We shall agree to write the compound statement (4) in the following manner.

$$\begin{cases} ax + by + c = 0 \\ a'x + b'y + c' = 0. \end{cases} \qquad \text{...(5)}$$

In the present section, we shall study the solution of the compund statement (4). We say that we solve the equations (1) and (2) *simultaneously* because any solution of (4) will be a solution of (1) and (2) simultaneously. The study of the solution of (4) is sometimes also referred to as a study of the solution of the simultaneous equations. But before doing so in general, we shall, in the following, discuss some examples.

Example. 1. Find the truth set of

$$2x - 3y + 4 = 0 \text{ and } 3x + y - 5 = 0.$$

Solution. We have already seen in the previous section that the ordered pairs

$$\left(0, \frac{4}{3}\right), \left(1, 2\right), \left(2, \frac{8}{3}\right), \left(-1, \frac{2}{3}\right), (-2, 0)$$

are some of the solutions of the first of these equations. The second equation is equivalent to

$$y = 5 - 3x.$$

By giving to x, different values,

$$0, 1, 2, -1, -2, \ldots\ldots$$

we get the corresponding values of y as

$$5, 2, -1, 8, 11, \ldots\ldots$$

so that some of the solutions of the second equation are

$$(0, 5), (1, 2), (2, -1), (-1, 8), (-2, 11).$$

We have an ordered pair (1 , 2) which is common to the truth sets of the two equations and so (1 , 2) is a solution of the two simultaneous equations. But it is quite possible that we had not tried the value 1 for x. Also, even if we have found a solution of the two equations, where is the guarantee that this is the only solution of the two equations. In the following, we give a more satisfactory method of solving the same problem.

Alternative Solution

We have

$$2x - 3y + 4 = 0 \quad \text{and} \quad 3x + y - 5 = 0$$

$$\Leftrightarrow \; 2x - 3y + 4 = 0 \quad \text{and} \quad 3(3x + y - 5) = 0.$$

(We have multiplied both sides of the second equation by 3.)

$$\Leftrightarrow \; 2x - 3y + 4 = 0 \quad \text{and} \quad 2x - 3y + 4 + 3\,(3x + y - 5) = 0.$$

(We have added to the second equation the first)

$$\Leftrightarrow \; 2x - 3y + 4 = 0 \quad \text{and} \quad 11x - 11 = 0$$

$$\Leftrightarrow \; 2x - 3y + 4 = 0 \quad \text{and} \quad x = 1$$

$$\Leftrightarrow \; 2\cdot 1 - 3y + 4 = 0 \quad \text{and} \quad x = 1$$

(We have used $x = 1$ in the first equation)

$$\Leftrightarrow \; 6 - 3y = 0 \quad \text{and} \quad x = 1$$

$$\Leftrightarrow \; y = 2 \quad \text{and} \quad x = 1$$

Now, the truth set of

$$y = 2 \quad \text{and} \quad x = 1,$$

is $\{(1, 2)\}$.

Thus, the truth set of the given *compound statement* is

$$\{(1, 2)\}.$$

In the present case, therefore, we get a unique solution of the two simultaneous equations.

Note. All that we have done is that first we reduce the given compound statement to an equivalent form in such a way that one of the two equations involves only one variable, Then, through necessary steps we arrive at an equivalent form

$$x = h \text{ and } y = k,$$

whose truth set is, as is obvious,

$$\{(h, k)\}.$$

2. Solve the equations

$$\begin{cases} 3x + 4y + 5 = 0 \\ 2x + 3y - 7 = 0 \end{cases}$$

Solution. We have

$$\begin{cases}3x+4y+5=0\\2x+3y-7=0\end{cases}$$
$$\Leftrightarrow\begin{cases}3(3x+4y+5)=0\\4(2x+3y-7)=0.\end{cases}$$

(We have multiplied the two equations by 3 and 4 respectively.)

$$\Leftrightarrow\begin{cases}3(3x+4y+5)=0\\4(2x+3y-7)-3(3x+4y+5)=0.\end{cases}$$

(We have subtracted the first from the second equation.)

$$\Leftrightarrow\begin{cases}3x+4y+5=0\\(8-9)\,x+(12-12)\,y-28-15=0\end{cases}$$
$$\Leftrightarrow\begin{cases}3x+4y+5=0\\-x-43=0\end{cases}$$
$$\Leftrightarrow\begin{cases}3x+4y+5=0\\x=-43\end{cases}$$
$$\Leftrightarrow\begin{cases}3(-43)+4y+5=0\\x=-43.\end{cases}$$

(We have used the second equation for changing the first.)

$$\Leftrightarrow\begin{cases}4y-124=0\\x=-43\end{cases}$$
$$\Leftrightarrow\begin{cases}y=31\\x=-43.\end{cases}$$

The required truth set, therefore, is

$$\{(-43, 31)\}.$$

Note. From the two examples discussed above, it may seem that a compound statement involving two linear equations in two variables joined with the connective 'and' will always have a unique solution. This observation is not correct. In fact, in the following, we have an example of each case

(*i*) where there is no solution,

(*ii*) where there are infinite number of solutions.

3. Find the truth set of

$$7x-2y+5=0 \quad\text{and}\quad 21x-6y+10=0.$$

Solution. We have

$$7x-2y+5=0 \quad\text{and}\quad 21x-6y+10=0$$
$$\Leftrightarrow\ 3(7x-2y+5)=0 \quad\text{and}\quad 21x-6y+10=0$$
$$\Leftrightarrow\ 3(7x-2y+5)=0$$
$$\text{and}\quad 21x-6y+10-3(7x-2y+5)=0$$
$$\Leftrightarrow\ 7x-2y+5=0 \quad\text{and}\quad 10-15=0$$

But $\qquad 10 - 15 = 0 \qquad$ is false.

Therefore, $7x - 2y + 5 = 0$ and $10 - 15 = 0$ is also false.

We have that the open statement

$$7x - 2y + 5 = 0 \text{ and } 21x - 6y + 10 = 0$$

is false. The required truth set is therefore *void*.

4. Find the truth set of

$$6x - 8y + 5 = 0 \text{ and } 9x - 12y + \frac{15}{2} = 0.$$

Solution. We have

$$\begin{aligned}
& 6x - 8y + 5 = 0 \text{ and } 9x - 12y + \frac{15}{2} = 0 \\
\Leftrightarrow\ & 3(6x - 8y + 5) = 0 \text{ and } 2\left(9x - 12y + \frac{15}{2}\right) = 0 \\
\Leftrightarrow\ & 3(6x - 8y + 5) = 0 \\
& \text{and } 2\left(9x - 12y + \frac{15}{2}\right) - 3(6x - 8y + 5) = 0 \\
\Leftrightarrow\ & 6x - 8y + 5 = 0 \text{ and } 0 = 0.
\end{aligned}$$

But $0 = 0$ is true. Therefore,

$$\begin{aligned}
& 6x - 8y + 5 = 0 \text{ and } 9x - 12y + \frac{15}{2} = 0 \\
\Leftrightarrow\ & 6x - 8y + 5 = 0 \\
\Leftrightarrow\ & 8y = 6x + 5 \\
\Leftrightarrow\ & y = \frac{6x + 5}{8}.
\end{aligned}$$

The truth set, therefore, is

$$\{(h, k) : k = \frac{6h + 5}{8}, h, k \in Q\}.$$

In view of the four examples that we have discussed above, we see that the truth set of the compound statement

$$ax + by + c = 0 \text{ and } a'x + b'y + c' = 0$$

is

either one elementic or void or infinite. Accordingly, we say that the simultaneous linear equations have (*i*) a unique solution, (*ii*) no solution and (*iii*) infinite number of solutions, respectively. Sometimes we also make the following statements in the three different cases.

(*i*) 'The linear equations are consistent.'

(*ii*) 'The linear equations are inconsistent.'

(*iii*) 'The linear equations are dependent.'

EXERCISES

1. Solve the following systems of equations.

(i) $\begin{cases} x + y = 25 \\ x - y = 4 \end{cases}$ (ii) $\begin{cases} 2x + y = 13 \\ 7x - y = 2 \end{cases}$

(iii) $\begin{cases} x + y = 11 \\ -x + y = 15 \end{cases}$ (iv) $\begin{cases} x + 3y = 7 \\ -x + 8y = 4 \end{cases}$

(v) $\begin{cases} x + 3y = 12 \\ 2x + 3y = 6 \end{cases}$ (vi) $\begin{cases} 4x - 5y = 2 \\ -4x + 11y = 10 \end{cases}$

(vii) $\begin{cases} 5x + 7y = 28 \\ 3x + 7y = 25 \end{cases}$ (viii) $\begin{cases} -3x + 4y = 13 \\ -3x - 2y = 10 \end{cases}$

(ix) $\begin{cases} 17x - 11y = 13 \\ 5x - 11y = 1 . \end{cases}$

2. Find the truth sets of the following systems of equations.

(i)	$2x - 5y + 6 = 0$	and	$3x + 4 = 0$
(ii)	$-3x + 4y - 2 = 0$	and	$3 - 2y = 0$
(iii)	$7x + 3y - 11 = 0$	and	$3x + 7y - 5 = 0$
(iv)	$2x + 3y - 15 = 0$	and	$3x - 2y + 4 = 0$
(v)	$2x + 4y - 7 = 0$	and	$6x + 8y - 9 = 0$
(vi)	$2x - 3y + 4 = 0$	and	$8x - 12y + 16 = 0$
(vii)	$6x - 21y + 12 = 0$	and	$10x - 35y + 20 = 0$
(viii)	$3x - 7y + 3 = 0$	and	$5x - 6y + 5 = 0$
(ix)	$4x + 5y - 5 = 0$	and	$7x + 8y - 8 = 0$
(x)	$8x - 14y + 10 = 0$	and	$12x - 21y + 14 = 0$
(xi)	$10x - 5y + 15 = 0$	and	$4x - 2y + 6 = 0$

3. Assuming the domain of the variables x and y to be Q_0 the set of zero rational numbers, find the truth sets of the following systems of equations

(i) $\dfrac{1}{x} + \dfrac{1}{y} = 12$, $\dfrac{1}{x} - \dfrac{1}{y} = 4$ (ii) $\dfrac{3}{x} + \dfrac{2}{y} = 5$, $\dfrac{5}{x} - \dfrac{2}{y} = 3$

(iii) $\dfrac{8}{x} + \dfrac{15}{y} = \dfrac{33}{2}$, $\dfrac{4}{x} - \dfrac{35}{y} = -\dfrac{43}{2}$ (iv) $\dfrac{1}{7x} + \dfrac{1}{6y} = 3$, $\dfrac{1}{2x} - \dfrac{1}{3y} = 5$

(v) $\dfrac{3}{4x} - \dfrac{3}{y} = \dfrac{7}{5}$, $\dfrac{5}{2x} + \dfrac{5}{2y} = -\dfrac{11}{3}$ (vi) $\dfrac{1}{x} + y = 3$, $\dfrac{3}{x} - y = 3$

(vii) $\frac{4}{x} + 5y = 7$

$\frac{3}{x} + 4y = 5$

(viii) $2x + \frac{3}{y} = 10$

$7x - \frac{5}{y} = 4$

(ix) $\frac{7}{x} - \frac{5}{y} = 12$

$\frac{14}{x} - \frac{10}{y} = 11$

(x) $\frac{14}{x} + \frac{7}{y} = 10$

$\frac{21}{x} + \frac{21}{2y} = 15$

4. Find the truth sets of the following systems of equations.

(i) $\frac{x}{4} - \frac{y + 32}{8} = 6$

$\frac{3x - 2y}{5} + \frac{y}{8} = 25$

(ii) $\frac{x}{20} + \frac{2y - 3}{11} = 4$

$3 - \frac{x}{5} + \frac{5y}{18} = 4$

(iii) $x + y = \frac{1}{7}(10x + y)$

$10x + y = (10y + x) + 18$

(iv) $\frac{x + y}{3} - \frac{x - y}{2} = 42$

$\frac{x - y}{9} + \frac{x}{2} = 5$

(v) $\frac{x - y}{4} - \frac{x - 3y}{5} = y - 3$

$\frac{3}{4}(x - y) + \frac{5}{6}(x + y) = 18$

(vi) $\frac{x - y}{10} + \frac{5x}{6} = 3$

$2x + 4y - \frac{2x + 3y}{2} = -2$

(vii) $\frac{x + y}{7} + \frac{x - y}{5} = 2$

$\frac{x + y}{7} - \frac{x - y}{5} = -2$

(viii) $\frac{2x + 3y}{5} + \frac{2x - 3y}{3} = 4$

$\frac{2x + 3y}{5} - \frac{2x - 3y}{3} = -4.$

Two Linear Equations in Two Variables. General Considerations.

In this section, we shall obtain conditions for the two equations

$$ax + by + c = 0$$
$$a'x + b'y + c' = 0$$

to be *consistent*, *inconsistent* or *dependent*. These conditions will be formulated as statements involving the co-efficients

$$a, b, c.\ ;\ a', b', c'.$$

We reduce the system to an equivalent form in which one of the equations involves-only one variable. This is then used to put the system in such an equivalent form that the solution becomes obvious.

In fact, we have

$$ax + by + c = 0 \text{ and } a'x + b'y + c' = 0$$
$$\Leftrightarrow \quad a'(ax + by + c) = 0 \text{ and } a(a'x + b'y + c') = 0.$$

(We have multiplied both sides of the two equations by a' and a respectively.)

$\Leftrightarrow$ $$a'(ax + by + c) = 0$$

and $$a(a'x + b'y + c') - a'(ax + by + c) = 0.$$

(We have subtracted from the second, the first.)

$\Leftrightarrow$ $$ax + by + c = 0$$

and $$(ab' - a'b)\,y + (ac' - a'c) = 0. \qquad \ldots(1)$$

Now, let us suppose that $$ab' \neq a'b.$$

Then $$ab' - a'b \neq 0$$

and so $$1/(ab' - a'b)$$

exists. The given system is then equivalent to

$$ax + by + c = 0$$

and $$y + \frac{ac' - a'c}{ab' - a'b} = 0$$

which again is equivalent to

$$ax + by + c = 0$$

and $$y = \frac{ca' - ac'}{ab' - a'b}$$

$\Leftrightarrow$ $$ax + b\,\frac{ca' - ac'}{ab' - a'b} + c = 0$$

and $$y = \frac{ca' - ac'}{ab' - a'b}$$

$\Leftrightarrow$ $$a(ab' - a'b)\,x + b(ca' - ac') + c(ab' - a'b) = 0$$

and $$y = \frac{ca' - ac'}{ab' - a'b}$$

$\Leftrightarrow$ $$a(ab' - a'b)\,x + a(b'c - bc') = 0$$

and $$y = \frac{ca' - ac'}{ab' - a'b}$$

$\Leftrightarrow$ $$x = \frac{bc' - b'c}{ab' - a'b}$$

and $$y = \frac{ca' - ac'}{ab' - a'b}.$$

The given system, therefore, has a unique solution

$$\left(\frac{bc' - cb'}{ab' - a'b}, \frac{ca' - ac'}{ab' - a'b}\right)$$

provided $$ab' - a'b \neq 0.$$

If, however, $$ab' - a'b = 0,$$

we have from (1) that the given system is equivalent to

$$ax + by + c = 0$$

and $$ac' - a'c = 0.$$

Further suppose that $$ac' - a'c \neq 0.$$

Then the statement $$ac' - a'c = 0$$

will be false, so that the compound statement

$$ax + by + c = 0$$

and

$$ac' - a'c = 0$$

will be false. The truth set of the given system will be *vaid.*

On the other hand if

$$ab' - a'b = 0$$

and

$$ac' - a'c = 0,$$

we shall have

$$ax + by + c = 0$$

and

$$ac' - a'c = 0$$

$$\Leftrightarrow ax + by + c = 0$$

because $ac' - a'c = 0$ is true.

Thus, the given system is equivalent to the single equation

$$ax + by + c = 0,$$

so that the truth set of the given system will be infinite.

The system of equations, therefore, is

(*i*) consistent (*ii*) inconsistent (*iii*) dependent

according as

(*i*) $ab' \neq a'b$ (*ii*) $ab' = a'b,\ ac' \neq a'c$

(*iii*) $ab' = a'b,\ ac' = a'c$

Working Rule. In case $ab' \neq a'b$, *i.e.*, when a unique solution of the system exists, we have the following rule to put down the solution.

First we write the detached co-efficients

$$\begin{array}{ccc} a & b & c \\ a' & b' & c'. \end{array}$$

We suppress the columns of co-efficients of x, y and the constant terms to obtain respectively

$$\begin{array}{cc} b & c \\ b' & c' \end{array} \qquad \begin{array}{cc} a & c \\ a' & c' \end{array} \qquad \begin{array}{cc} a & b \\ a' & b'. \end{array}$$

In between we put the arrows as indicated above. We agree to write

$$bc' - bc' = \begin{array}{cc} b & c \\ b' & c' \end{array}$$

$$ca' - ac' = \begin{array}{cc} a & c \\ a' & c' \end{array}$$

$$ab' - a'b = \begin{array}{cc} a & b \\ a' & b'. \end{array}$$

The solution of the system can be exhibited in the following manner.

$$\left\{ \frac{\begin{matrix} b & c \\ b' & c' \end{matrix}}{\begin{matrix} a & b \\ a' & b' \end{matrix}}, \frac{\begin{matrix} a & c \\ a' & c' \end{matrix}}{\begin{matrix} a & b \\ a' & b' \end{matrix}} \right\}$$

This rule for writing the unique solution is referred to as the *cross multiplication* rule. Using this rule we solve an example below.

Example. Solve the system

$$\begin{cases} 3x - 5y + 4 = 0 \\ 12y + 4x - 3 = 0 \end{cases}$$

Solution. We rewrite the system in the form

$$\begin{cases} 3x + (-5)\,y + 4 = 0 \\ 4x + 12y + (-3) = 0. \end{cases}$$

Detaching the co-efficients we have

$$\begin{matrix} 3 & -5 & 4 \\ 4 & 12 & -3. \end{matrix}$$

The required solution, then is

$$\left\{ \frac{\begin{matrix} -5 & 4 \\ 12 & -3 \end{matrix}}{\begin{matrix} 3 & -5 \\ 4 & 12 \end{matrix}}, \frac{\begin{matrix} 3 & 4 \\ 4 & -3 \end{matrix}}{\begin{matrix} 3 & -5 \\ 4 & 12 \end{matrix}} \right\}$$

i.e.,
$$\left(\frac{(-5)(-3) - 12 \times 4}{3 \times 12 - 4(-5)}, \frac{4 \times 4 - 3 \times (-3)}{3 \times 12 - 4(-5)}\right)$$

or
$$\left(\frac{15 - 48}{36 + 20}, \frac{16 + 9}{36 + 20}\right)$$

or
$$\left(\frac{-33}{56}, \frac{25}{56}\right)$$

EXERCISES

1. Assuming a, b, d to be different non-zero rational numbers, solve the following systems of equations.

(i) $\begin{cases} ax + by + c = 0 \\ ax - by + d = 0 \end{cases}$ (ii) $\begin{cases} ax + by + c = 0 \\ -ax + by + c = 0 \end{cases}$

(iii) $\begin{cases} ax + by + c = 0 \\ bx - ay + d = 0 \end{cases}$ (iv) $\begin{cases} ax + by + c = 0 \\ -bx + ay + d = 0 \end{cases}$

(v) $\begin{cases} ax + by + c = 0 \\ dx + by + e = 0 \end{cases}$ (vi) $\begin{cases} ax + by + c = 0 \\ ax + dy + e = 0 \end{cases}$

2. Find the truth sets of the following systems of equations by the *cross multiplication rule.*

(*i*) $\begin{cases} 2x + 3y - 11 = 0 \\ 2x - 3y + 7 = 0 \end{cases}$ (*ii*) $\begin{cases} 2x + 3y - 5 = 0 \\ -2x + 3y - 7 = 0 \end{cases}$

(*iii*) $\begin{cases} 2x + 3y - 11 = 0 \\ 3x + 2y - 7 = 0 \end{cases}$ (*iv*) $\begin{cases} 2x + 3y - 11 = 0 \\ 3x - 2y + 7 = 0 \end{cases}$

(*v*) $\begin{cases} 2x + 3y - 7 = 0 \\ -3x + 2y + 5 = 0 \end{cases}$ (*vi*) $\begin{cases} 3x + 4y + 8 = 0 \\ 5x + 2y - 4 = 0 \end{cases}$

(*vii*) $\begin{cases} 4y - 3x - 11 = 0 \\ 7x - 3y + 5 = 0 \end{cases}$ (*viii*) $\begin{cases} 7x - 11y + 5 = 0 \\ 3y - 5x - 3 = 0. \end{cases}$

Consistency of Three Linear Equations in Two Variables

Consider the three equations

$$ax + ay + c = 0 \qquad \ldots(1)$$

$$a'x + b'y + c' = 0 \qquad \ldots(2)$$

$$a''x + b''y + c'' = 0. \qquad \ldots(3)$$

The system of three equations is consistent. if there exists an ordered pair (h, k) of rational numbers which is a solution of all the three equations. A discussion of the condition of consistency in the general case being beyond the scope of the present work, we assume that two of the three equations admit of a unique solution. We suppose that the first two equations have a unique solution and then find the condition that the system is consistent. Thus, we work under the assumption

$$ab' \neq a'b.$$

In such a case a unique solution of (1) and (2) is

$$\left(\frac{bc' - b'c}{ab' - a'b}, \frac{ca' - ac'}{ab' - a'b}\right).$$

It will also be a solution of (3) provided

$$a'' \frac{bc' - b'c}{ac' - a'b} + b'' \frac{ca' - ac'}{ab' - a'b} + c'' = 0$$

or equivalently,

$$a''(bc' - b'c) + b''(ca' - ac') + c''(ab' - a'b) = 0. \qquad \ldots(4)$$

Thus, (4) is the condition of consistency of the given system under the assumption $ab' \neq a'b$. It may, however, be remarked here that the condition will turn out to be the same if we assume the pairs (*ii*), (*iii*) or (*iii*), (*i*) to have a unique solution.

The condition (4) is also referred to as the *Eliminant* of the given system of equations and the process of finding the condition of consistency is called *Elimination.*

EXERCISE

Which of the following systems of equations are consistent and which are not ?

(i) $\begin{cases} x + y - 3 = 0 \\ 2x + 3y - 8 = 0 \\ 5x + 8y - 11 = 0 \end{cases}$ (ii) $\begin{cases} 2x - 3y + 1 = 0 \\ 5x + 8y - 13 = 0 \\ 12x + 13y + 5 = 0 \end{cases}$

(iii) $\begin{cases} 5x + 3y - 13 = 0 \\ 2y - 5x - 8 = 0 \\ 7x + 4y + 18 = 0 \end{cases}$ (iv) $\begin{cases} 4x - 11y - 1 = 0 \\ 7x + 5y - 26 = 0 \\ 2x + y - 7 = 0. \end{cases}$

46. LINEAR EQUATION IN THREE VARIABLES.

An equation in three variables x, y, z is seid to be linear, if it is equivalent to an equation of the form

$$ax + by + cz + d = 0 \qquad (1)$$

where a, b, c, d are rational numbers, a, b, c being not all zero.

For example the equations

(i) $x + y + z = 5$ (ii) $(7x - 3) + (y - 4) = (\frac{1}{2}z + 5)$

are linear, inasmuch as they are equivalent to

(iii) $1 . x + 1 . y + 1 . z + (-5) = 0$

(iv) $7x + 1 . y + (-\frac{1}{2}) z + (-12) = 0$

respectively, which are of the form (1) above.

Solution of a Linear Equation in Three Variables

Consider the linear equation

$$2x + 5y - 7z - 3 = 0$$

where the domain of each of the three variables x, y, z is the set Q of rational numbers. The equation is equivalent to

$$2x + 5y - 3 = 7z$$

$$\Leftrightarrow \frac{2x + 5y - 3}{7} = z.$$

By giving to x and y any values as members of Q, we obtain a corresponding value of z also in Q as for example if x and y take the values 0, 0, the value of z is $-\frac{3}{7}$.

We say then, the *ordered triplet* $\left[0, 0, -\frac{3}{7}\right]$ is a solution of the given equation. Similarly, we may see that

$$\left(0, 1, \frac{2}{7}\right), \left(1, 0, \frac{-1}{7}\right), \left(1, 1, \frac{4}{7}\right), \left(2, 1, \frac{6}{7}\right)$$

are some other ordered triplets of rational numbers which are solutions of the

equation. Of course not every ordered triplet is a solution of the equation. For example, as may be easily verified, the ordered triplet

$$\left(0, 1, \frac{-1}{7}\right)$$

is *not* a solution of the given equation. We have put down five ordered triplets of rational numbers which are solutions of the given equations. Because any values could be given to x and y and then we can find out the corresponding value of z, we see that the truth set of the given equation, consisting of ordered triplets of rational numbers, is infinite.

The set $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ *or* $\mathbf{Q}^3$

Definition. *The set* $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ *or* $\mathbf{Q}^3$ *is defined as the set of all ordered triplets* (a, b, c), *where* $a, b, c \in \mathbf{Q}$ *The numbers* a, b, c *are called the first, the second and the third members respectively of the ordered triplet* (a, b, c).

For example,

$$(1, 1, 1), (0, 0, 1), \left(1, \frac{1}{2}, \frac{1}{3}\right), (1, \cdot 01, \cdot 001)$$

are some members of the set $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$.

In the set builder notation, $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ can be symbolically described in the following manner :

$$\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} = \{(a, b, c) : a \in \mathbf{Q}, b \in \mathbf{Q}, c \in \mathbf{Q}\}.$$

Definition. *Two ordered triplets are said to be equal, the same or identical if and only if their first, second and third members are respectively equal.*

Thus, we have

$$(a, b, c) = (d, e, f) \Leftrightarrow a = d, b = e \text{ and } c = f.$$

EXERCISES

1. Which of the following equations are linear and which are not ? State the restrictions you have to impose on the variables x, y and z in each case.

(*i*) $3x - 2y + 4z - 11 = (x - 5) + (2 - 3y) + (14z + 7)$

(*ii*) $\dfrac{x + 2y}{4} + \dfrac{z - 3}{5} = \dfrac{2x - 32}{10} + \dfrac{y}{20}$

(*iii*) $\dfrac{x - 3y + 4}{2z - 5} + 5 = 0$ (*iv*) $\dfrac{5x + 11y + 13z - 5}{7 - 12y} + 3 = 0$

(*v*) $\dfrac{-2x + 4y + z}{5x - 7} + \dfrac{5}{2} = 0$ (*vi*) $\dfrac{3x - 7y + 5z + 22}{y - 2} + \dfrac{x}{z - 3} = 0$

(*vii*) $\dfrac{x - 5y + z}{3y - 8} = 0$ (*viii*) $\dfrac{x - 3}{y - 2} + \dfrac{3z}{2y - 4} + 5 = 0.$

2. State whether or not the two ordered triplets are the same.

(*i*) (1, 2, 3), (3 / 3, 6 / 3, 9 / 3) (*ii*) (1, 2, 3), (2, 3, 1)

(*iii*) (1, 2, 3), (1 + 1, 2 + 1, 3 + 1) (*iv*) (1, 2, 3), (1, 4, 3)

(*v*) $(a, b, c), (-a, -b, -c)$ (*vi*) $(a, b, c), (a^2, b^2, c^2)$

(*vii*) $(a, b, c), (a + d, b + d, c + a)$ (*viii*) $(a, b, c). (a, -b, c)$

(*ix*) $(a, b, c), (ad, bd, cd)$ (*x*) $(a, b, c), (a / d, b / d, c / d)$,

[In (*v*) to (*x*) it is assumed that a, b, c, d are different non-zero rational numbers.]

3. Describe the sets of ordered triplets for which the following expressions are not meaningful.

(*i*) $\dfrac{3x - 4y + 3}{z}$ (*ii*) $\dfrac{x + y + z - 5}{3x}$

(*iii*) $\dfrac{x + y - 3z}{y}$ (*iv*) $\dfrac{y + z}{3 - 5x}$

(*v*) $\dfrac{y - 2}{x - 5} + \dfrac{3z + 5}{2y + 3}$. (*vi*) $\dfrac{1}{z + 5} - \dfrac{3x + 5}{y - 4}$.

4 Find at least three solutions of each of the following equations.

(*i*) $x + y + z + 1 = 0$ (*ii*) $3x - 2y + 4z - 11 = 0$

(*iii*) $(2x - 5) + (y + 3) + (7 - 3z) = 0$

(*iv*) $2x - 4y + 3z = x - y + 5z + 7$.

5. Find at least two solutions of each of the linear equations of Ex. 1 above.

47. TWO LINEAR EQUATIONS IN THREE VARIABLES.

We consider two linear equations

$$ax + by + cz + d = 0 \qquad \ldots(1)$$

$$a'x + b'y + c'z + d' = 0 \qquad \ldots(2)$$

where $a, b, c, d : a', b', c', d'$ are all rational numbers and the domain of each of the variables x, y, z is the set Q. In the present section we study the compound statement

$$ax + by + cz + d = 0 \text{ and } a'x + b'y + c'z + d' = 0$$

which we agree to write in the form

$$\begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0. \end{cases}$$

With the help of examples we shall see that such a system can have an infinite number of solutions or no solution depending upon the values of a, b, c, etc,

Examples. 1, Find the truth set of

$$\begin{cases} x - 2y + 5z + 11 = 0 \\ 3x + 4y - 7z + 3 = 0. \end{cases}$$

Solution. The given system is equivalent to

$$\begin{cases} x - 2y + 5z + 11 = 0 \\ 3x + 4y - 7z + 3 - 3(x - 2y + 5z + 11) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 5z + 11 = 0 \\ 10y - 22z - 30 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 5z + 11 = 0 \\ 5y - 11z - 15 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 5z + 11 = 0 \\ y = \dfrac{11z + 15}{5}. \end{cases}$$

Now, by giving to z any arbitrary value c as a member of **Q**, we find the value of y as $b = \dfrac{11c + 15}{5}$.

From the first of the two equations then we find the value a of x as

$$a = 2b - 5c - 11,$$

The solution set is, therefore, infinite. In fact it is

$$\left\{\left(a, b, c \right) : b = \frac{11c + 5}{5}, a = 2b - 5c - 11, c \in Q \right\}.$$

2. Find the truth set of

$$\begin{cases} 3x - 6y + 9z + 4 = 0 \\ 4x - 8y + 12z + 5 = 0. \end{cases}$$

Solution. The given system is equivalent to

$$\begin{cases} 4(3x - 6y + 9z + 4) = 0 \\ 3(4x - 8y + 12z + 5) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 4(3x - 6y + 9z + 4) = 0 \\ 3(4x - 8y + 12z + 5) - 4(3x - 6y + 9z + 4) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 4(3x - 6y + 9z + 4) = 0 \\ 15 - 16 = 0. \end{cases}$$

But $15 - 16 = 0$ is false, so that the compound statement

$$4(3x - 6y + 9z + 4) = 0 \text{ and } 15 - 16 = 0$$

is false. Therefore, the truth set of the given system is empty. We say that the equations are inconsistent.

EXERCISES

1. Find at least two solutions of the following systems of equations.

(*i*) $\begin{cases} x + y + z + 5 = 0 \\ 2x - y + 3z - 7 = 0 \end{cases}$ (*ii*) $\begin{cases} 2x - 5y - 3z - 11 = 0 \\ 3x + 8y - z + 4 = 0 \end{cases}$

(*iii*) $\begin{cases} 5x - 2y + 3z - 5 = 0 \\ 3x + 4y - 3z + 2 = 0 \end{cases}$ (*iv*) $\begin{cases} 4x - y + 3z = 0 \\ 3x + y - 3z + 5 = 0 \end{cases}$

(*v*) $\begin{cases} 15x - 6y + 9z + 5 = 0 \\ 20x - 8y + 12z + \frac{20}{3} = 0 \end{cases}$ (*vi*) $\begin{cases} 6x + 3y - 9z + 12 = 0 \\ 4x + 2y - 9z + 8 = 0. \end{cases}$

2. Show that the truth set of each of the following systems of equations is void.

(i) $\begin{cases} x + y - z + 3 = 0 \\ 3x + 3y - 3z + 7 = 0 \end{cases}$ (ii) $\begin{cases} 2x - 3y + 5z - 8 = 0 \\ 6x - 9y + 15z + 5 = 0 \end{cases}$

48. THREE LINEAR EQUATIONS IN THREE VARIABLES.

Consider three linear equations in x, y and z

$$ax + by + cz + d = 0 \qquad ...(1)$$
$$a'x + b'y + c'z + d' = 0 \qquad ...(2)$$
$$a''x + b''y + c''z + d'' = 0 \qquad ...(3)$$

where $a, b, c, d\,;\, a', b', c', d'\,;\, a'', b'', c'', d''$ are all rational numbers and the domain of each of the variables x, y, z is the set Q. We shall study the solution of the compound statement

$ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ and $a''x + b''y + c''z + d'' = 0$.

We agree to write this compound statement in the form

$$\begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ a''x + b''y + c''z + d'' = 0. \end{cases}$$

The general case becomes pretty involved and so we do not study the solution of the general case. However, through examples, we shall see that the system of the equations can have, depending upon the values of a, b, c, d, etc., a unique solution, an infinite number of solutions or no solution.

Examples. 1. Find the truth set of

$$\begin{cases} x + y + z + 3 = 0 \\ x + 2y + 3z + 6 = 0 \\ x + 3y + 6z + 10 = 0. \end{cases}$$

Solution. The given system is equivalent to

$$\begin{cases} x + y + z + 3 = 0 \\ x + 2y + 3z + 6 - (x + y + z + 3) = 0 \\ x + 3y + 6z + 10 - (x + 2y + 3z + 6) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z + 3 = 0 \\ y + 2z + 3 = 0 \\ y + 3z + 4 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z + 3 = 0 \\ y + 2z + 3 = 0 \\ y + 3z + 4 - (y + 2z + 3) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z + 3 = 0 \\ y + 2z + 3 = 0 \\ z + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z \quad + 3 = 0 \\ y + 2(-1) + 3 = 0 \\ \qquad\quad z + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z + 3 = 0 \\ \qquad y + 1 = 3 \\ \qquad z + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + (-1) + (-1) + 3 = 0 \\ \qquad\qquad\qquad y + 1 = 0 \\ \qquad\qquad\qquad z + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + 1 = 0 \\ y + 1 = 0 \\ z + 1 = 0. \end{cases}$$

The truth set of the given system of equations is

$$\{(-1, -1, -1)\}.$$

2. Find the truth set of

$$\begin{cases} x + y + z - 10 = 0 \\ 2x + 3y + 4z - 33 = 0 \\ 3x + 5y + 7z - 56 = 0. \end{cases}$$

Solution. The given system equations is equivalent to

$$\begin{cases} x + y + z - 10 = 0 \\ 2x + 3y + 4z - 33 - 2(x + y + z - 10) = 0 \\ 3x + 5y + 7z - 56 - 3(x + y + z - 10) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z - 10 = 0 \\ \quad y + 2z - 13 = 0 \\ \quad 2y + 4z - 26 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \qquad\qquad x + y + z - 10 = 0 \\ \qquad\qquad\quad y + 2z - 13 = 0 \\ 2y + 4z - 26 - 2(y + 2z - 13) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z - 10 = 0 \\ \quad y + 2z - 13 = 0 \\ \qquad\qquad 0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - y + z - 10 = 0 \\ \quad y + 2z - 13 = 0 \end{cases}$$

as $0 = 0$ is true.

Now, by giving to z any value as a member of $\mathbf{Q}$, we can find a value of y from the second of the above two equations. These values when substituted in the first equation give a value of x. For example, if z is giving the value 0, y will be 13

and x will be -3, so that $(-3, 13, 0)$ will be a solution of the given system. The truth set will be

$$\{(a, b, c) : a = c - 3,\ b = 13 - 2c,\ a, b, c \in Q\}.$$

3. Find the truth set of

$$\begin{cases} x - 2y + z + 1 = 0 \\ x - 8y + 3z + 7 = 0 \\ 2x - y + z + 1 = 0. \end{cases}$$

Solution. The system is equivalent to

$$\begin{cases} x - 2y + z + 1 = 0 \\ x - 8y + 3z + 7 - (x - 2y + z + 1) = 0 \\ 2x - y + z + 1 - 2(x - 2y + z + 1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + z + 1 = 0 \\ -6y + 2z + 6 = 0 \\ 3y - z - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + z + 1 = 0 \\ 3y - z - 3 = 0 \\ 3y - z - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + z + 1 = 0 \\ 3y - z - 3 = 0 \\ 3y - z - 3 - (3y - z - 1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + z + 1 = 0 \\ 3y - z - 3 = 0 \\ -2 = 0. \end{cases}$$

Because $-2 = 0$ is false, we see that the required truth set is void.

Note. We see that the system of equations may have

(*i*) unique solution

(*ii*) infinite number of solutions

(*iii*) no solution.

Correspondingly, we say that the system is

(*i*) consistent, (*ii*) dependent (*iii*) inconsistent.

EXERCISES

1. Find the truth set of the following systems of equations.

(i) $\begin{cases} 4x - 5y + 6z - 3 = 0 \\ 8x - 7y + 3z + 3 = 0 \\ 7x - 8y + 9z + 6 = 0 \end{cases}$

(*ii*) $\begin{cases} x + y = 35 \\ y + z = 37 \\ z + x = 42 \end{cases}$

(*iii*) $\begin{cases} 2x - 5y + 6z + 41 = 0 \\ 5x - 3y + 2z + 22 = 0 \\ 3x - 6y + 4z + 37 = 0 \end{cases}$

(*iv*) $\begin{cases} 3x + 2y = 34 \\ 3y + 2z = 44 \\ 3z + 2x = 42 \end{cases}$

(v) $\begin{cases} 2x - 3y + z + 1 = 0 \\ 5x - 6y + 3z + 13 = 0 \\ x + z + 11 = 0 \end{cases}$ (vi) $\begin{cases} 2x + 3y + 6z + 1 = 0 \\ 5x + 2y - z + 4 = 0 \\ x + 7y + 19z + 1 = 0 \end{cases}$

(vii) $\begin{cases} 2x + 3y + 4z + 4 = 0 \\ 3x - 2y - 5z + 10 = 0 \\ 5x + 14y + 21z + 6 = 0 \end{cases}$ (viii) $\begin{cases} x + y - z + 7 = 0 \\ 3x - 8y + 7z + 5 = 0 \\ 22y - 50z + 25 = 0 \end{cases}$

2. Assuming the domains of the variables x, y, z to be the set Q_0 of non-zero rational numbers find the truth sets of the following systems of equations.

(i) $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} + 6 = 0 \\ \dfrac{2}{x} + \dfrac{3}{y} + \dfrac{4}{z} + 8 = 0 \\ \dfrac{3}{x} + \dfrac{4}{y} + \dfrac{6}{z} + 10 = 0 \end{cases}$ (ii) $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 3 \\ \dfrac{1}{y} + \dfrac{1}{z} = 5 \\ \dfrac{1}{z} + \dfrac{1}{x} = 6 \end{cases}$

(iii) $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} + 5 = 0 \\ \dfrac{2}{y} + \dfrac{3}{z} + 6 = 0 \\ \dfrac{3}{z} + \dfrac{1}{x} + 5 = 0 \end{cases}$ (iv) $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} - \dfrac{1}{z} = 1 \\ \dfrac{3}{x} - \dfrac{4}{y} + \dfrac{5}{z} = 9 \\ \dfrac{2}{x} + \dfrac{3}{y} + \dfrac{4}{z} = 19. \end{cases}$

49. PROBLEMS.

In this section, we shall see how our knowledge of the solution of systems of linear equations can be made use for solving problems of *Arithmetic*. We exhibit this by taking an example each from

(*i*) numbers (*ii*) time and work (*iii*) profit and loss (*iv*) time and distance (*v*) stocks and shares.

Examples. 1. The sum of the digits of a number consisting of two digits is 14. If the digits are reversed, the number diminishes by 18 Find the number.

Solution. Let the unit's digit be x and the ten's digit be y.

Then $x + y = 14.$ ...(1)

Also, the number is $x + 10\,y.$

If the digits are reversed, the number becomes

$$10x + y.$$

Also $(x + 10y) - (10x + y) = 18.$ ...(2)

We obtain, therefore, the system of equations (1) and (2), which we have to solve for x and y to get the result.

In fact, we have the above system equivalent to

$$\begin{cases} x + y - 14 = 0 \\ -x + y - 2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y - 14 = 0 \\ (-x + y - 2) + (x + y - 14) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y - 14 = 0 \\ 2y - 16 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y - 14 = 0 \\ y = 8 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + 8 - 14 = 0 \\ y = 8 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 6 \\ y = 8. \end{cases}$$

The required number is 86.

2. Three men and four boys can do a piece of work in five days and one man and 16 boys can do it in four days. In how many days can one man and four boys do this work ?

Solution. Let us suppose that a man alone can complete the work in x days and a boy in y days. Certainly x and y are positive rational numbers.

Then 3 men and 4 boys will finish

$$\frac{3}{x} + \frac{4}{y}$$

of the work in one day. As they take 5 days to complete the work, we have,

$$5\left(\frac{3}{x} + \frac{4}{y}\right) = 1. \qquad \ldots(1)$$

Similarly, because one man and 16 boys finish the wor. in 4 days, we have

$$4\left(\frac{1}{x} + \frac{16}{y}\right) = 1. \qquad \ldots(2)$$

The system of equations (1) and (2) is equivalent to

$$\begin{cases} \frac{3}{x} + \frac{4}{y} - \frac{1}{5} = 0 \\ \frac{1}{x} + \frac{16}{y} - \frac{1}{4} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(\frac{3}{x} + \frac{4}{y} - \frac{1}{5}\right) - 3\left(\frac{1}{x} + \frac{16}{y} - \frac{1}{4}\right) = 0 \\ \frac{1}{x} + \frac{16}{y} - \frac{1}{4} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -\frac{44}{y} + \frac{11}{20} = 0 \\ \frac{1}{x} + \frac{16}{y} - \frac{1}{4} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{1}{y} = \dfrac{1}{80} \\ \dfrac{1}{x} + \dfrac{16}{80} - \dfrac{1}{4} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{1}{y} = \dfrac{1}{80} \\ \dfrac{1}{x} = \dfrac{1}{20}. \end{cases}$$

Now, one man and four boys will be able to complete

$$\frac{1}{20} + \frac{4}{80}$$

of the work in one day.

If z denotes the number of days they will need to complete the work, we have

$$z\left(\frac{1}{20} + \frac{4}{80}\right) = 1$$

which is equivalent to $\qquad z = 10.$

3. A horse and a cow were sold for Rs. 760 making a profit of 25% on the horse and 10% on the cow. By selling them for Rs. 767·50, the profit realised would have been 10% on the horse and 25% on the cow. Find the cost price of each.

Solution. Let the cost price of the horse and the cow be x and y rupees respectively.

Their selling prices in rupees in the first case will be

$$\frac{125}{100}x \text{ i.e., } \frac{5}{4}x \quad \text{and} \quad \frac{110}{100}y \text{ i.e., } \frac{11}{10}y$$

respectively, so that we have

$$\frac{5}{4}x + \frac{11}{10}y = 760. \qquad \ldots (1)$$

Similarly, we have

$$\frac{11}{10}x + \frac{5}{4}y = 767{\cdot}5. \qquad \ldots (2)$$

Now, the system of equations (1) and (2) is equivalent to

$$\begin{cases} 25x + 22y - 15200 = 0 \\ 22x + 25y - 15350 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 22\,(25x + 22y - 15200) = 0 \\ 25\,(22x + 25y - 15350) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 22\,(25x + 22y - 15200) - 25\,(22x + 25y - 15350) = 0 \\ 22x + 25y - 15350 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -141y + 49350 = 0 \\ 22x + 25y - 15350 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 350 \\ 22x + 25y - 15350 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 350 \\ 22x + 25 \times 350 - 15350 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 350 \\ x = 300. \end{cases}$$

The cost price of the horse is Rs. 300 and that of the cow is Rs. 350.

4. A boat goes upstream 30 km. and downstream 44 km in 10 hours. It also goes upstream 40 km. and downstream 55 km. in 13 hours. Find the speed of the stream and that of the boat in still water.

Solution. Let u and v denote the speeds of the boat in still water and the stream, respectively in km. per hour.

Then the speed of the boat while going upstream will be $(u - v)$ km. per hour and while going downstream it will be $(u + v)$ km. per hour.

The total time taken in the first case being 10 hours, we have

$$\frac{30}{u-v} + \frac{44}{u+v} = 10. \qquad \ldots(1)$$

Similarly,

$$\frac{40}{u-v} + \frac{55}{u+v} = 13. \qquad \ldots(2)$$

The system of equations (1) and (2) is equivalent to

$$\Leftrightarrow \begin{cases} 4\left(\dfrac{30}{u-v} + \dfrac{44}{u+v} - 10\right) = 0 \\ 3\left(\dfrac{40}{u-v} + \dfrac{55}{u+v} - 13\right) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 4\left(\dfrac{30}{u-v} + \dfrac{44}{u+v} - 10\right) = 0 \\ 3\left(\dfrac{40}{u-v} + \dfrac{55}{u+v} - 13\right) \\ -4\left(\dfrac{30}{u-v} + \dfrac{44}{u+v} - 10\right) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{30}{u-v} + \dfrac{44}{u+v} - 10 = 0 \\ -\dfrac{11}{u+v} + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{30}{u-v} + \dfrac{44}{u+v} - 10 = 0 \\ u + v = 11 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{30}{u-v} + \dfrac{44}{11} - 10 = 0 \\ u + v = 11 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{30}{u-v} - 6 = 0 \\ u + v = 11 \end{cases}$$

$$\Leftrightarrow \begin{cases} u - v = 5 \\ u + v = 11 \end{cases}$$

$$\Leftrightarrow \begin{cases} (u + v) + (u - v) = 5 + 11 \\ u + v = 11 \end{cases}$$

$$\Leftrightarrow \begin{cases} u = 8 \\ u + v = 11 \end{cases}$$

$$\Leftrightarrow \begin{cases} u = 11 \\ v = 3. \end{cases}$$

Thus, the speed of the boat in still water is 8 km. per hour and that of the stream is 3 km. per hour.

5. A man invested Rs. 6,200 partly in 10% stock at 132 and partly in 8% stock at 99. If the income derived from each investment be the same, find the two investments.

Solution. By the statement 10% stock at 132, we mean that in order to purchase a stock whose value is Rs. 100, we have to pay Rs. 132 and then the income from this investment of Rs. 132 will be Rs 10 *p.a.* Similarly, by the statement '8% stock at 99' we mean that by investing Rs. 99, we get stock worth Rs. 100 and the income will be Rs. 8 *p.a.*

Suppose now that the man invested Rs. x and Rs. y respectively in the two stocks.

His total investment being Rs. 6,200, we have

$$x + y = 6200. \qquad (1)$$

Again his income from the two investments being the same, we have

$$\frac{x}{132} \times 10 = \frac{y}{99} \times 8. \qquad \ldots(2)$$

The system of equations (1) and (2) is equivalent to

$$\Leftrightarrow \begin{cases} x + y = 6200. \\ 45x - 48y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y - 6200 = 0 \\ 15x - 16y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y - 6200 = 0 \\ 15x - 16y - 15\,(x + y - 6200) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y - 6200 = 0 \\ -\,31x + 15 \times 6200 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y - 6200 = 0 \\ y = 3000 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + 3000 - 6200 = 0 \\ y = 3000 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 3200 \\ y = 3000 \end{cases}$$

The two investments must be Rs. 3,200 and Rs. 3,000.

EXERCISES

1. A number consists of two digits whose sum is 8. Find the number, if by adding 18 to it, the digits are reversed.

2. A number consists of two digits whose sum is one-fourth of the number. If the digits are reversed the number becomes 27 more then the original number. Find the number.

3. A number consists of three digits whose sum is 17 ; the middle digit exceeds the sum of the other two by '1'. If the digits be reversed the number is diminished by 396. Find the number.

4. Five years hence father's age will be three times that of the son and five years ago father was seven times as old as the son. Find their present ages.

5. A man has five sons, the sum of the ages of the five sons being equal to the age of the father. In 12 years' time the sum of the ages of the sons will be double that of their father. What is the age of the father now ?

6. Three men and four boys can do a piece of work in five days and two men and twelve boys can do it in four days. In how many days will one man and two boys do this work ?

7. Three men and nine boys together can do four times as much work as a man and a boy together can do in the same time. Find the ratio of the work done by a man and a boy respectively in the same time.

8. Four copies of the Algebra book and five copies of the Geometry book cost Rs. 49, while seven copies of the Algebra book and 4 copies of the Geometry book cost Rs. 62. Find the price of each.

9. A man sold nine horses and seven cows to one person for Rs. 12,000, and six horses and thirteen cows to another person for the same amount. What was the price of each ?

10. A kg. of tea and 3 kg. of sugar cost Rs. 19·50. If the price of sugar rose by 50% and tea by 10%, they would cost Rs. 23·25. Find the prices per kg. of tea and sugar.

11. A train 75 metres long overtook a person who was running at the rate of 8 km. an hour and passed him in 7·5 seconds. Subsequently it overtook a second person and passed him in 6·75 seconds. At what rate was the second person walking ?

12. The current of a stream runs at the rate of five km. an hour. A motor boat goes 10 km upstream and back again to its starting point in 50 minutes. Find the speed of the motor boat in still water.

13. Anil and Ajay run a kilometre. First, Anil gives Ajay a start of 25 metres and beats him by 51 seconds. Secondly, Anil gives Ajay a start of 1 minute 15 seconds and is beaten by 50 metres. Find the times in which Anil and Ajay run a kilometre.

14. Navin and Sunil cycle from A to B a distance of 55 km. Navin arrives 30 minutes before Sunil. They then cycle from B to A. By giving to Sunil a start of 4 km , Navin arrives 6 minutes before him. Find the speeds of each of them in km., *p.h.*

15. Sushil walks a certain distance at a certain rate. Had he walked $\frac{1}{2}$ km. an hour faster, he would have taken 15 minutes less. But if he had walked 1 km. an hour slower, he would have taken 45 minutes more. Find the distance and Sushil's speed.

16. Ram and Shyam have the same income. Ram saves one-fifth of his income. But Shyam by spending Rs. 1,000 annually more than Ram, finds himself in a debt of Rs. 2,000 at the end of four years. What was the annual income of each ?

17. The sum of Rs, 1,550 was lent partly at 7·5% and partly at 12% simple interest. The total interest received after three years was Rs. 450. How much was lent in each case ?

18. A man invests Rs 6,750 partly in 12% at 140 and partly in 10% at 125. If his total income is Rs. 560, how much has he invested in each ?

19. A man invested the same sum in two different stocks 7% at $103\frac{1}{2}$% and 8% at 105. His income from one is Rs. 186 more than from the other. What was his investment in each ?

20. A man wishes to invest Rs. 21,000 between two stocks 15% at 143 and $10\frac{1}{2}$% at 91, so as to derive the same income from each. How must he do this ?

SUMMARY

Assuming the domain of each of the variables occuring in equations to be the set **Q** and the co-efficients in the equations as members of **Q**, we have the following.

(1) The linear equation

$$ax + b = 3, a \neq 0,$$

has a unique solution *viz.,*

$$-\frac{b}{a}.$$

(2) The equations

$$\begin{cases} ax + b = 0 & a \neq 0 \\ cx + d = 0 & c \neq 0 \end{cases}$$

are consistent if and only if

$$ad = bc.$$

(3) The truth set of the equation

$$ax + by + c = 0$$

a and b being not both zero, is infinite.

(4) The system

$$\begin{cases} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{cases}$$

has (*i*) a unique solution (*ii*) an infinite number of solutions (*iii*) no solution according as

(*i*) $ab' \neq a'b$ (*ii*) $ab' = a'b,\ ac' = a'c$ (*iii*) $ab' = a'b,\ ac' \neq a'c$.

Also accordingly we say that the system is

(*i*) consistent (*ii*) dependent (*iii*) inconsistent.

(5) The truth set of

$$ax + by + cz + d = 0$$

a, b, c being not all zero, is infinite.

(6) The truth set of the system

$$\begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}$$

may be infinite or *void*.

(7) The system

$$\begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ a''x + b''y + c''z + d'' = 0 \end{cases}$$

may have

(*i*) a unique solution (*ii*) no solution (*iii*) an infinite number of solutions depending upon the values of the co-efficients.

REVIEW EXERCISES

1. Solve the following systems of equations :

(*i*) $\begin{cases} 13x + 12y - 13 = 0 \\ 12x + 13y - 12 = 0 \end{cases}$ (*ii*) $\begin{cases} 5x + 4y - 22 = 0 \\ 4x - 5y + 7 = 0 \end{cases}$

(*iii*) $\begin{cases} 7x - 11y + 3 = 0 \\ 2x + 5y - 8 = 0 \end{cases}$ (*iv*) $\begin{cases} 2x - 4y + 8 = 0 \\ 3x - 6y + 9 = 0 \end{cases}$

(*v*) $\begin{cases} 21x + 25y - 13 = 0 \\ 4x - 3y + 5 = 0 \end{cases}$ (*vi*) $\begin{cases} 4x - 6y + 12 = 0 \\ 15y - 10x - 30 = 0. \end{cases}$

2. Which of the following systems of equations are consistent and which are not ?

(i) $\begin{cases} 2x - 3y + 5 = 0 \\ 7x + 2y - 9 = 0 \\ x + 36y - 77 = 0 \end{cases}$ (ii) $\begin{cases} x + y + 11 = 0 \\ 2x + 4y - 14 = 0 \\ 2x + 5y + 12 = 0 \end{cases}$

(iii) $\begin{cases} 3x - 4y + 13 = 0 \\ 4x - 2y + 5 = 0 \\ 22x + 31y - 7 = 0 \end{cases}$ (iv) $\begin{cases} 5x + 4y - 3 = 0 \\ 2x - 5y + 4 = 0 \\ 3x - 24y + 19 = 0 \end{cases}$

3. Assuming the domain of each of the variables to be the set Q_0 of non-zero rational numbers, solve the following systems of equations.

(i) $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = 5 \\ \dfrac{5}{x} + \dfrac{2}{3} = 3 \end{cases}$ (ii) $\begin{cases} \dfrac{4}{3x} + \dfrac{5}{4y} = \dfrac{1}{2} \\ \dfrac{1}{2x} + \dfrac{7}{3y} = \dfrac{3}{5}. \end{cases}$

4. Assuming $a \neq b$, find the condition of consistency of

$$\begin{cases} x + y = 1 \\ ax + by = c \\ ax^2 + b^2y = c^2. \end{cases}$$

5. Solve the following systems of equations.

(i) $\begin{cases} x - 2y + 3z + 4 = 0 \\ 2x + 5y - 8z - 7 = 0 \\ 5x + 23y - 36z - 37 = 0 \end{cases}$ (ii) $\begin{cases} 2x - 5y + 3z - 8 = 0 \\ 4x + 3y - 11z + 5 = 0 \\ 13y - 17z - 21 = 0 \end{cases}$

(iii) $\begin{cases} 2x = 3 \\ 3x - 7y = 10 \\ 9x + 8y - 7z = 2 \end{cases}$ (iv) $\begin{cases} -3x + 4y - 5z = 0 \\ x + 2y - 9z + 4 = 0 \\ 13x - 7y - z + 11 = 0. \end{cases}$

6. Assuming the domain of the variables to be Q_0, solve the following systems of equations.

(i) $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} - 4 = 0 \\ \dfrac{1}{y} + \dfrac{1}{z} - 6 = 0 \\ \dfrac{1}{z} + \dfrac{1}{x} - 8 = 0 \end{cases}$ (ii) $\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} - 5 = 0 \\ \dfrac{3}{y} + \dfrac{4}{z} - 6 = 0 \\ \dfrac{4}{z} + \dfrac{2}{x} - 5 = 0 \end{cases}$

7. A number consists of two digits whose sum is 10. If the digits are reversed the number becomes 36 more than what it was. Find the number.

8. Asha tells Usha "Give me Rs. 900 and I shall have twice as much as you will have". Usha replies 'If you give me Rs. 100, I shall have thrice as much as you will have'. How much money has each got ?

9. A fraction becomes 3/4 if 1 is added to its numerator. However, if it is multiplied by 2/5, it becomes 1/4. Find the fraction.

10. A and B working together can finish a job in $1\frac{4}{5}$ days. The job can also be done if A works for $2\frac{1}{2}$ days and B for $\frac{3}{4}$ day. How long will it take each working alone to do the job.

11. If three taps are opened together, a cistern is filled in 12 hours. One of the taps can fill it in 10 hours and another in 15 hours. How does the third tap work ?

12. A dealer has two cars which together cost him Rs. 30,000. He sells one at a profit of 20% and the other at a profit of 8%. If he gains 15% on the whole, find the cost price of each car.

13. A man purchases a certain number of oranges at 3 for 50 paise and some others of a different kind at 2 for 25 paise. Thus, he paid Rs. 36 in all. He put the oranges together, and sold all but 16 of them at 20 paise each. He made a profit of Rs. 8.80. How many oranges of each kind did he buy ?

14. The age of the father is 3 years more than three times the age of the son. Three years hence father's age will be 10 years more than twice the age of the son. Find their ages now.

15. The sum of the present ages of Mohan and Sohan is 63 years. Also, Mohan is twice as old as Sohan was when Mohan was as old as Sohan is. Find their ages.

16. A boat covers a distance of 6 km. in 45 minutes while going downstream. But in coming back upstream it takes $1\frac{1}{2}$ hours. Find the speed of the current and the speed of the boat in still water.

17. A journey of 600 km. is covered partly by train and partly by car. It takes 8 hours if 120 km. is covered by train and the rest by car, but it takes 20 minutes longer if 200 km. is covered by train and the rest by car. Find the speeds of the train and the car.

18. There are two solutions, of strengths 8% and 24% of iodine. Find the quantity of each solution to be mixed up in order to obtain 80 c.c. of a 12% solution.

19. A person invests Rs. 14,970 partly in 6% stock at 90 and partly in $6\frac{1}{2}$% at 97. Find how much of each stock he bought, if his total income is Rs. 1,000.

20. Having Rs. 8,370 to invest, a man puts part of it in 9% stock at 96 and the remaining in 12% stock at 120. Find the amount invested in each, if his income from the two investments is the same.

6

Quadratic Equations

50. INTRODUCTION

As in chapter 5, in this chapter also, unless otherwise stated, the variables will be thought of as having the set Q of rational numbers as the domain and the numbers involved will also be rational.

Let us consider algebraic expressions of the form

$$2x + 3\ ,\ 3xy^2\ ,\ xy + x\ ,\ 3x^2 - 2x + 5\ ,\ x^2 + 3xy + y^2,$$
$$3x^4 + 2x^3 - 3x + 7.$$

Each one of these expressions is formed by a finite number of addition and multiplication *operations* with rational numbers and variables. These expressions are examples of what are called **polynomials.**

Definition. *A polynomial over Q is an algebraic expression formed by a finite number of addition and multiplication operations with rational numbers and variables. The index of any variable in a polynomial must be a non-negative integer.*

For example, while

$$2x^2 + 3{\cdot}5x + y\ ,\ 3x^3 - 5x^2 + 4x + 7$$

are polynomials,

$$\frac{x}{y} + 3$$

is not a polynomial inasmuch as in x/y the index of the variable y is -1 which is a negative integer.

The simplest form of a polynomial is a *monomial* which is either a numeral, or a variable, or a product of a numeral and one or more variables.

Thus,

$$5x^2\ ,\ -3{\cdot}5x\ ,\ 7xy^3$$

are some examples of monomials. A polynomial may, therefore, be thought of as the sum of a finite number of monomials. Each one of the monomials constituting the sum as a polynomial, is called a *term* of the polynomial. If the number of terms in a polynomial is two, it is called a *binomial* and if the number of terms is three, we call it a *trinomial.*

EXERCISE

Identify each of the following polynomials as a monomial, binomial, trinomial.

(*i*) $x + 1$ (*ii*) $8x^2-5x + 1$

(*iii*) $\frac{12}{5} xy$ (*iv*) $7 + 3.4$

(*v*) $3(x + y)$ (*vi*) $8xy + y + 2x$

(*vii*) 3 (*viii*) $5x^2 + 4$

(*ix*) $x + y + z$.

In the present chapter, however, we shall be concerned with polynomial in one variable only.

Definition. *A polynomial in one variable x over* **Q** *is an algebraic expression of the form.*

$$a^0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a^r x^{n-r} + \ldots\ldots + a_{n-1} x + a_n$$

where $a^0 , a_1 , \ldots\ a^r , \ldots\ldots , a_n$

are given rational numbers, $a^0 \neq 0$, n is any natural number and the domain of x is **Q**. The numbers $a_0 , a_1 , a_2 , \ldots\ldots , a_{n-1}$ are called the co-efficients of

$$x^n , x^{n-1} , x^{n-2} , \ldots\ldots , x$$

respectively, a_n is called the constant term and n is called the degree of the polynomial. Also, the polynomial is called *monic* if the co-efficient of the highest degree term is 1.

For example,

(*i*) $2x + 5$ (*ii*) $7x^2 + 5x - 3$ (*iii*) $x^3 - 2x + 1$

are polynomials in x of degree one, two and three respectively. Of these (*iii*) is monic but (*i*) and (*ii*) are not monic.

A polynomial of degree one or a first degree polynomial is also known as a *linear polynomial.* Also a polynomial of degree two or a second degree polynomial is called a *quadratic polynomial.*

It may be noted that in our study of linear equations and linear inequalities with one unknown, we were concerned with linear polynomials. The general form of a linear polynomial with the variable x is

$$ax + b$$

where a and b are given rational numbers and $a \neq 0$. We have seen that a linear equation with one unknown x is an open statement which is equivalent to that of the form

$$ax + b = 0,$$

a, b being given rational numbers and $a \neq 0$. Also a linear inequality with one unknown is an open statement which is equivalent to one having any of the following forms ·

(*i*) $ax + b > 0$ (*ii*) $ax + b < 0$

(*iii*) $ax + b \geqslant 0$ (*iv*) $ax + b \leqslant 0$.

Here again a, b are given rational numbers and $a \neq 0$.

In the present chapter, we shall deal with open statements which are equivalent to those of the form

(*i*) $ax^2 + bx + c = 0$ (*ii*) $ax^2 + bx + c > 0$

(*iii*) $ax^2 + bx + c < 0$ (*iv*) $ax^2 + bx + c \geqslant 0$

(*v*) $ax^2 + bx + c \leqslant 0$,

a, b, c being given rational numbers and $a \neq 0$.

Thus, in this chapter, we shall be concerned with quadratic polynomials of the form

$$ax^2 + bx + c$$

where a, b, c are any given numbers and $a \neq 0$, and x is the variable with the set **Q** of rational numbers as its domain.

Of course instead of denoting a variable by letter x, we may denote the variable by any other letter say y, u, v, t.

Note. It is possible that the co-efficient of x or the constant term in the quardiatic polynomial is zero.

EXERCISES

1. Give the co-efficients including the constant terms of the following polynomials. Also state the degree of each one of them.

(*i*) $2x + 7$ (*ii*) $-2x + 5$

(*iii*) $-3x$ (*iv*) $\cdot 5y + \cdot 7$

(*v*) $-1\cdot 5y + 2$ (*vi*) $7y$

(*vii*) $-2x^2 + 7$ (*viii*) $\cdot 5x^2 - 7x + 8$

(*ix*) $\cdot 7x^2 + x$ (*x*) $3y^2 - 2y + 1$

(*xi*) $\frac{5}{2}y^2 + y - 3$ (*xii*) $-8y^2$

(*xiii*) $t^2 + t - 1$ (*xiv*) $-2t^2 + \frac{5}{3}t$

(*xv*) $3t^2 - 7$ (*xvi*) $x^3 + 7x^2 - 3x + 5$

(*vii*) $2x^4 - 2x + 5$ (*xviii*) $x^7 - 1$.

2. Which of the polynomials in Ex. 8 above are monic ?

3. Put down any five quadratic polynomials and the co-efficients including the constant term in each case. Which of these are monic ?

4. Put down any two polynomials of degree three which are monic. State the co-efficients including the constant term in each case.

5. Put down a polynomial of degree five. Also put down the co-efficients including the constant term.

51. PRODUCT OF TWO LINEAR POLYNOMIALS.

It is important to note that *the product of two linear polynomials with the same variable is a quadratic polynomial.*

Consider two linear polynomials

$$ax + b,\ cx + d;\ a \neq 0,\ c \neq 0.$$

Repeatedly using the distributive law, the commutative and the associative laws of addition and multiplication, we have

$$\begin{aligned}(ax + b)(cx + d) &= ax(cx + d) + b(cx + d)\\ &= axcx + axd + bcx + bd\\ &= acx^2 + (ad + bc)\,x + bd.\end{aligned}$$

Also we note that

$$a \neq 0 \quad \text{and} \quad c \neq 0 \quad \Rightarrow \quad ac \neq 0.$$

The product, therefore, is a quadratic polynomial in that the co-efficient ac of the term acx^2 is not zero.

Note. We should note that the equality

$$(ax + b)(cx + d) = acx^2 + (ad + bc)\,x + bd$$

is true $\forall\ x \in Q$.

EXERCISES

1. Express the following products of linear polynomials as quadratic polynomials.

(*i*) $(x + 1)(x + 2)$ (*ii*) $(x - 2)(x + 3)$

(*iii*) $(x - 4)(x - 7)$ (*iv*) $(2x + 1)(x + 2)$

(*v*) $(2x + 3)(3x + 2)$ (*vi*) $(5x - 7)(6x + 11)$

(*vii*) $(x + 4)(3x - 1)$ (*viii*) $(-2x + 1)(x - 7)$

(*ix*) $(3x + 4)(x - 2)$ (*x*) $(2 - 3t)(3t + 1)$

(*xi*) $(5 - 4t)(5 + 4t)$ (*xii*) $(2 - t)(2t + 5)$.

2. a, b, p, q, l, m being any rational numbers, express the following products as quadratic polynomials :

(*i*) $(x + a)(x + b)$ (*ii*) $(x - 2p)(x + 3q)$ (*iii*) $(y + l)(y - 5m)$

3. Put down any five pairs of linear polynomials with the same variable and obtain their product as a quadratic polynomial in each case.

Square of a Monic Linear Polynomial

In Ex. 1 and 2 above we may see that the product of two linear polynomials which are monic is a monic quadratic polynomial. As a special case of the product of two linear polynomials, let us consider the square of a monic linear polynomial. A linear polynomial which is monic, is of the form

$$x + p$$

where p is any rational number. We have

$$\begin{aligned}(x + p)^2 &= (x + p)(x + p)\\ &= x(x + p) + p(x + p)\\ &= (x^2 + xp) + (px + p^2)\\ &= x^2 + 2px + p^2\end{aligned}$$

We notice that the constant term is p^2 and the co-efficient of x is $2p$. Thus, the square of a monic linear polynomial is a monic quadratic polynomial such that *the constant term is the square of half the co-efficient of* x.

Conversely, every monic quadratic polynomial such that the constant term is the square of half the co-efficient of x is the square of a monic linear polynomial.

For example, each of the following is a monic quadratic polynomial satisfying this condition,

(*i*) $x^2 + 4x + 4$ (*ii*) $x^2 + 2bx + b^2$

(*iii*) $x^2 + 3x + \frac{9}{4}$ (*iv*) $x^2 - 5x + \frac{25}{4}$

(*v*) $x^2 - lx + \frac{1}{4} l^2$ (*vi*) $x^2 - 6x + 9$

and they are respectively the squares of the following linear polynomials which are also monic.

(*i*) $x + 2$ (*ii*) $x + b$

(*iii*) $x + \frac{3}{2}$ (*iv*) $x - \frac{5}{2}$

(*v*) $x - \frac{1}{2}$ (*vi*) $x - 3$.

Now, let us consider the monic quadratic polynomial

$$x^2 + lx + \ldots$$

whose constant term is not known. This constant term is uniquely determined in

order that the polynomial be the square of a linear polynomial. Thus, the missing term being the square of half of the co-efficient of x, will be

$$\left(\frac{l}{2}\right)^2, \text{ i.e., } \frac{l^2}{4}$$

Also with this as the constant term, the polynomial will be the square of

$$x + \frac{l}{2}$$

so that we have

$$x^2 + lx + \frac{l^2}{4} = \left(x + \frac{l}{2}\right)^2.$$

EXERCISES

1. Each of the following is the square of a linear polynomial. Supply the missing terms.

(i) $x^2 - 4x + \ldots$ (ii) $x^2 + 3x + \ldots$

(iii) $x^2 - 2ax + \ldots$ (iv) $x^2 - x + \ldots$

(v) $x^2 - 5x + \ldots$ (vi) $x^2 + \frac{b}{a}x + \ldots$

(vii) $x^2 - \frac{1}{2}x + \ldots$ (viii) $x^2 + \frac{7}{4}x + \ldots$

(ix) $x^2 - \frac{9}{11}x + \ldots$

Also give the corresponding linear polynomial in each case

2. Each of the following is the square of a linear polynominal. What is the number k in each case ? l, m are given rational numbers.

(i) $x^2 + 2x + (2 + k)$ (ii) $x^2 - 3x + (7 - k)$

(iii) $x^2 + 6x + (\frac{9}{3} + k)$ (iv) $x^2 + 5x + (2 + k)$

(v) $x^2 + lx + (m + k)$ (vi) $x^2 + 2lx + (m - k)$.

3. Each of the following is the square of a linear polynominal. Supply the missing terms and give the corresponding linear polynomial in each case.

(i) $x^2 + \ldots + 9$ (ii) $x^2 + \ldots + \frac{9}{4}$.

4. Find k such that each of the following becomes the square of a linear polynomial ; l, m are given rational numbers.

(i) $x^2 + (2 + k)x + 4$ (ii) $x^2 + (5 - k)x + \frac{9}{25}$

(iii) $x^2 + (l + k)x + m^2$ (iv) $x^2 + (k - m)x + \frac{l^2}{4}$.

52. LINEAR FACTORS OF A QUADRATIC POLYNOMIAL.

Having seen that the product of two linear polynomials with the same variable is a quadratic polynomial, we now attend to the converse problem of expressing a given quadratic polynomial as a product of two linear polynomials.

It will be seen that *not* every quadratic polynomial with rational co-efficients can be expressed as a product of two linear polynomials with rational co-efficients. In fact we shall need to extend the system of rational numbers to that of real numbers and complex numbers to be able to express every quadratic polynomial as a product of linear polynomials and this programme of extension will engage our attention in Algebra II.

We shall now proceed to obtain the conditions to be satisfied by a quadratic polynomial with rational co-efficients to be expressible as a product of linear polynomials with rational co-efficients. Of course, we shall also learn to express those of the quadratic polynomials, which can be so expressed, as products of linear polynomials.

It will be seen that *the expression of a quadratic polynomial as a product of linear polynomials constitutes the technique for the determination of the truth sets of quadratic equations and inequalities.*

We say that the linear polynomial

$$lx + m, \qquad l, m \in \mathbf{Q}, l \neq 0$$

is a factor of the quadratic polynomial

$$ax^2 + bx + c, a, b, c \in \mathbf{Q}, a \neq 0$$

if there exists a linear polynomial

$$px + q, \qquad p, q \in \mathbf{Q}, p \neq 0$$

such that

$$ax^2 + bx + c = (lx + m)(px + q), \forall\ x \in \mathbf{Q}.$$

Condition for a Quadratic Polynomial to be Factorisable

Theorem. *The quadratic polynomial*

$$ax^2 + bx + c, \quad a, b, c \in \mathbf{Q}, a \neq 0$$

is expressible as a product of two linear polynomials with rational co-efficients if and only if

$$b^2 - 4ac$$

is the square of a rational number.

Proof. Let us suppose that

$$b^2 - 4ac$$

is the square of a rational number.

We have, a not being zero,

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$$

Now,

$$x^2 + \frac{b}{a}x + \frac{c}{a}$$

is monic with b/a as the co-efficient of x. Further, the square of half of this co-efficient of x is

$$\left(\frac{b}{2a}\right)^2, \text{ i.e., } \frac{b^2}{4a^2}.$$

We have,

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)$$

$$= \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

$$= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}.$$

We have assumed that $b^2 - 4ac$ is the square of a rational number. Let this rational number be k, so that we have

$$k^2 = b^2 - 4ac.$$

Thus, we have

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{k^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{k}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right) + \frac{k}{2a}\right]\left[\left(x + \frac{b}{2a}\right) - \frac{k}{2a}\right]$$

$$= a\left(x + \frac{b+k}{2a}\right)\left(x + \frac{b-k}{2a}\right)$$

$$= \left(ax + \frac{b+k}{2}\right)\left(x + \frac{b-k}{2a}\right).$$

We have been able to show, therefore, that the quadratic polynomial

$$ax^2 + bx + c, \quad a, b, c \in Q, \quad a \neq 0$$

is expressible as a product of two linear factors with rational co-efficients, if

$$b^2 - 4ac$$

is the square of a rational number

Conversely, we shall now prove that if $ax^2 + bx + c$ is expressible as a product of two linear factors, then $b^2 - 4ac$ must be the square of a rational number.

Let $ax^2 + bx + c$ be the product of $lx + m$, $px + q$, so that we have

$$ax^2 + bx + c = (lx + m)(px + q).$$

Also, we have

$$(lx + m)(px + q) = lpx^2 + (lq + mp)\,x + mq,$$

so that

$$a = lp, \quad b = lq + mp, \quad c = mq.$$

This gives

$$\begin{aligned} b^2 - 4ac &= (lq + mp)^2 - 4lp\,mq \\ &= l^2q^2 + m^2p^2 + 2lmpq - 4lpmq \\ &= l^2q^2 + m^2p^2 - 2lmpq \\ &= (lq - mp)^2 \end{aligned}$$

so that $b^2 - 4ac$ is the square of the rational number $lq - mp$

Thus, the theorem is proved.

Illustration. Consider the quadratic polynomials :

(*i*) $6x^2 - 7x - 3$ (*ii*) $2x^2 + x - 5$

(*iii*) $12x^2 - x - 6$ (*iv*) $3x^2 + x + 8$

(*v*) $x^2 + 4$ (*vi*) $8x^2 + 10x - 3$

(*vii*) $4x^2 - 12x + 9$.

(*i*) $a = 6, \;\; b = -7, \;\; c = -3$.

$$b^2 - 4ac = (-7)^2 - 4(6)(-3) = 121 = 11^2$$

so that $b^2 - 4ac$ is the square of a rational number *viz*, 11.

(*ii*) $a = 2, \;\; b = 1, \;\; c = -5$.

$$b^2 - 4ac = 1^2 - 4(2)(-5) = 41$$

so that there is no rational number whose square is the rational number $b^2 - 4ac$ which is 41.

(*iii*) $a = 12, \;\; b = -1, \;\; c = -6$.

$$b^2 - 4ac = (-1)^2 - 4(12)(-6) = 289 = 17^2$$

so that $b^2 - 4ac$ is the square of a rational number 17.

(*iv*) $a = 3, \;\; b = 1, \;\; c = 8$.

$$b^2 - 4ac = 1^2 - 4(3)(8) = -95$$

so that $b^2 - 4ac$ is not the square of a rational number.

In fact, no negative rational number is the square of a rational number.

(*v*) $a = 1, \;\; b = 0, \;\; c = 4$

$$b^2 - 4ac = 0^2 - 4(1)(4) = -16$$

so that $b^2 - 4ac$ being negative, is not the square of a rational number.

(*vi*) $a = 8, \;\; b = 10, \;\; c = -3$.

$$b^2 - 4ac = (10)^2 - 4(8)(-3) = 196$$

so that $b^2 - 4ac$ is the square of a rational number 14.

(*vii*) $a = 4,\ b = -12,\ c = 9.$

$$b^2 - 4ac = (-12)^2 - 4(4)(9) = 0$$

and so $b^2 - 4ac$ is the square of a rational number *viz*, 0.

Thus, we see that the quadratic polynomials (*i*), (*iii*), (*vi*) and (*vii*) are expressible as products of linear factors with rational co-efficients, and the polynomials (*ii*), (*iv*) and (*v*) are not so expressible. In the following, we shall obtain the linear factors of the polynomials (*i*), (*iii*), (*vi*), (*vii*).

(*i*) We have

$$\begin{aligned}
6x^2 - 7x - 3 &= 6\left[x^2 - \frac{7}{6}x - \frac{3}{6}\right] \\
&= 6\left[\left\{x^2 - \frac{7}{6}x + \left(\frac{7}{12}\right)^2\right\} - \left\{\frac{3}{6} + \left(\frac{7}{12}\right)^2\right\}\right] \\
&= 6\left[\left(x - \frac{7}{12}\right)^2 - \left(\frac{3}{6} + \frac{49}{144}\right)\right] \\
&= 6\left[\left(x - \frac{7}{12}\right)^2 - \frac{121}{144}\right] \\
&= 6\left[\left(x - \frac{7}{12}\right)^2 - \left(\frac{11}{12}\right)^2\right] \\
&= 6\left[\left(x - \frac{7}{12}\right) + \frac{11}{12}\right]\left[\left(x - \frac{7}{12}\right) - \frac{11}{12}\right] \\
&= 6\left(x + \frac{4}{12}\right)\left(x - \frac{18}{12}\right) \\
&= 3.2\left(x + \frac{1}{3}\right)\left(x - \frac{3}{2}\right) \\
&= 3\left(x + \frac{1}{3}\right)2\left(x - \frac{3}{2}\right) \\
&= (3x + 1)(2x - 3).
\end{aligned}$$

(*iii*) We have

$$\begin{aligned}
12x^2 - x - 6 &= 12\left[x^2 - \frac{1}{12}x - \frac{6}{12}\right] \\
&= 12\left[\left\{x^2 - \frac{1}{12}x + \left(\frac{1}{24}\right)^2\right\} - \left\{\frac{6}{12} + \left(\frac{1}{24}\right)^2\right\}\right] \\
&= 12\left[\left(x - \frac{1}{24}\right)^2 - \frac{289}{576}\right] \\
&= 12\left[\left(x - \frac{1}{24}\right)^2 - \left(\frac{17}{24}\right)^2\right] \\
&= 12\left[\left(x - \frac{1}{24} + \frac{17}{24}\right)\left(x - \frac{1}{24} - \frac{17}{24}\right)\right]
\end{aligned}$$

$$= 12\left(x + \frac{16}{24}\right)\left(x - \frac{18}{24}\right)$$

$$= 4 \cdot 3\left(x + \frac{2}{3}\right)\left(x - \frac{3}{4}\right)$$

$$= 3\left(x + \frac{2}{3}\right)4\left(x - \frac{3}{4}\right)$$

$$= (3x + 2)(4x - 3).$$

We may similarly show that

(*vi*) $8x^2 + 10x - 3 = (4x - 1)(2x + 3)$

and

(*vii*) $4x^2 - 12x + 9 = (2x - 3)^2$.

EXERCISES

Which of the following quadratic polynomials are expressible as products of linear factors with rational co-efficients ? Express those, which are thus expressible, as products of linear factors.

(*i*) $x^2 + 5x + 6$

(*ii*) $x^2 - 9x + 8$

(*iii*) $x^2 + 4x + 7$

(*iv*) $x^2 + 2x - 8$

(*v*) $x^2 + 3x - 5$

(*vi*) $x^2 - 3x - 10$

(*vii*) $2x^2 - 7x + 5$

(*viii*) $3x^2 + 8x + 4$

(*ix*) $4x^2 - 9x + 6$

(*x*) $10x^2 - 23x - 5$

(*xi*) $6x^2 + 8x - 5$

(*xii*) $8x^2 + 13x - 6$

(*xiii*) $9x^2 + 24x + 16$

(*xiv*) $4x^2 - 9x + 4$

(*xv*) $7x^2 + 16x + 4$.

Note. The reader may note that the above method of expressing a quadratic polynomial as a product of linear factors becomes quite complicated so far as computations are concerned in cases where the co-efficients have comparatively large numerical values. In the following, we shall describe how linear factors can be easily obtained by inspection. Of course the condition for existence of these factors is satisfied.

Factorisation By Inspection

Let

$$ax^2 + bx + c, \qquad a, b, c \in \mathbf{Q}, a \neq 0$$

be such that it *can* be expressed as a product of two linear factors with rational co-efficients. We can always rewrite this polynomial in the form

$$\frac{1}{k}\left[kax^2 + kbx + kc\right]$$

where k is some suitable non-zero rational number such that

$$ka, kb, kc$$

are all members of the set I. The problem of writing

$$ax^2 + bx + c, \quad a, b, c \in \mathbf{Q}, \quad a \neq 0$$

as product of linear factors, then reduces to that of expressing

$$(ka)\,x^2 + (kb)\,x + kc, \quad ka, kb, bc \in \mathbf{I}, ka \neq 0$$

as a product of linear factors. We may, therefore, deal with the problem of expressing

$$ax^2 + bx + c, \quad a, b, c \in \mathbf{I}, a \neq 0$$

as a product of linear factors.

Let

$$ax^2 + bx + c = (lx + m)(px + q)$$

where, l, m, p, q are members of I.

Also, we have

$$(lx + m)(px + q) = lpx^2 + (lq + mp)\,x + mq$$

so that

$$a = lp, \quad b = lq + mp, \quad c = mq.$$

This gives

$$ac = (lp)(mq) = (lq)(mp).$$

Now $\quad a \in \mathbf{I}, c \in \mathbf{I} \Rightarrow ac \in \mathbf{I}.$

Also lq and mp are two factors of ac such that their sum is b. We have, therefore, the following procedure to get the factors of the quadratic polynomial.

Rule. Express the product ac of the co-efficient a of x^2 and the constant term c as the product of two integers in such a way that the sum of these two numbers is the co-efficient b of x. Express b as the sum of these two numbers and proceed employing the distributive law.

We illustrate the procedure with the help of the following examples.

Examples

Factorise the following.

(*i*) $4x^2 + 12x + 5$ (*ii*) $4x^2 - 23x - 6$

(*iii*) $6x^2 - 13x + 5$ (*iv*) $5x^2 + 13x - 6$.

(*i*) The product of the co-efficient of x^2 and the constant term is 4×5 *i.e*, 20.

Again, of the several pairs of factors of 20, we select the pair 10, 2 in that

$$10 + 2$$

We have

$$\begin{aligned}4x^2 + 12x + 5 &= 4x^2 + (10 + 2)\,x + 5\\ &= (4x^2 + 10x) + (2x + 5)\\ &= 2x\,(2x + 5) + 1\,.\,(2x + 5)\\ &= (2x + 1)\,(2x + 5).\end{aligned}$$

(*ii*) The product of the co-efficient of x^2 and the constant term is $4\,(-6)$ *i.e.*, -24.

Of the pairs of factors of -24, we select the pair -24, 1, as

$$-24 + 1$$

is the co-efficient -23 of x.

We have

$$\begin{aligned}4x^2 - 23x - 6 &= 4x^2 + (-24 + 1)\,x - 6\\ &= (4x^2 - 24x) + (x - 6)\\ &= 4x\,(x - 6) + 1\,.\,(x - 6)\\ &= (4x + 1)\,(x - 6).\end{aligned}$$

(*iii*) The product of the co-efficient of x^2 and the constant term is 6×5 *i.e.*, 30.

We choose the pair -10 and -3 of factors of 30 because their sum $-10 + (-3)$ is the eo-efficient -13 of x.

We have

$$\begin{aligned}6x^2 - 13x + 5 &= 6x^2 - (10 + 3)\,x + 5\\ &= (6x^2 - 10x) - (3x - 5)\\ &= 2x\,(3x - 5) + (-1)\,(3x - 5)\\ &= (2x - 1)\,(3x - 5).\end{aligned}$$

(*iv*) The product of the co-efficient of x^2 and the constant term is $5\,(-6)$ *i.e.*, -30.

We choose the pair 15, -2 of factors of -30 because their sum $15 - 2$ is the co-efficient -13 of x.

We have

$$\begin{aligned}5x^2 + 13x - 6 &= 5x^2 + (15 - 2)\,x - 6\\ &= (5x^2 + 15x) - (2x + 6)\\ &= 5x\,(x + 3) - 2(x + 3)\\ &= (5x - 2)\,(x + 3).\end{aligned}$$

EXERCISE

Factorise the following.

(*i*) $x^2 + 6x + 5$ (*ii*) $x^2 - 3x + 2$

(*iii*) $x^2 + 4x - 5$ (*iv*) $x^2 - 12x - 28$

(v) $2x^2 + 7x + 5$ (vi) $6x^2 - 13x + 6$

(vii) $3y^2 - 5y + 2$ (viii) $12y^2 - 11y - 15$

(ix) $2y^2 - 5y - 25$ (x) $12t^2 - 32t + 21$

(xi) $3t^2 + 7t + 2$ (xii) $3t^2 - 7t - 6$

(xiii) $14u^2 - 15u - 11$ (xiv) $21 - 4u - u^2$

(xv) $9u^2 - 30u + 25$.

53. THE QUADRATIC EQUATION $ax^2 + bx + c = 0$ OVER Q.

In this section we shall study the various methods of solving the quadratic equation

$$ax^2 + bx + c = 0,\ a, b, c \in Q,\ a \neq 0.$$

First of all we shall obtain the criterion for a given linear polynomial to be a factor of a given quadratic polynomial. This criterion gives us a relation between the roots of the equation and the factors of the corresponding polynomial in the form of the following theorem.

Theorem. *A rational number, h, is a root of the quadratic equation*

$$ax^2 + bx + c = 0 \quad a, b, c \in Q,\ a \neq 0$$

if and only if $x - h$ is a factor of $ax^2 + bx + c$.

Proof. Let a rational number h be a root of

$$ax^2 + bx + c = 0,$$

so that

$$ah^2 + bh + c = 0.$$

We have $\forall\ x \in Q$,

$$\begin{aligned} ax^2 + bx + c &= (ax^2 + bx + c) - 0 \\ &= (ax^2 + bx + c) - (ah^2 + bh + c) \\ &= a(x^2 - h^2) + b(x - h) \\ &= a(x - h)(x + h) + b(x - h) \\ &= (x - h)[a(x + h) + b] \\ &= (x - h)[ax + (ah + b)] \end{aligned}$$

so that $x - h$ is a factor of $ax^2 + bx + c$.

Conversely, let us suppose that $x - h$ is a factor of

$$ax^2 + bx + c,$$

so that there exists a linear polynomial $lx + m$ with rational co-efficients such that

$$ax^2 + bx + c = (x - h)(lx + m),\ \forall\ x \in Q.$$

Replacing x by h, we have

$$ah^2 + bh + c = (h - h)(lh + m) = 0\,(lh + m) = 0$$

so that h is a root of the quadratic equation

$$ax^2 + bx + c = 0.$$

Truth Set of the Quadratic Equation $ax^2 + bx + c = 0$.

We have already seen that

$$ax^2 + bx + c$$

is the product of two linear factors with rational co-efficients if and only if

$$b^2 - 4ac$$

is the square of a rational number.

Let $b^2 - 4ac$ *be the square of a rational number.*

Then we have a relation of the form

$$ax^2 + bx + c = (lx + m)(px + q). \qquad \ldots(1)$$

Now $\qquad lx + m = 0 \Leftrightarrow x = -m/l$

and $\qquad px + q = 0 \Leftrightarrow x = -q/p.$

Replacing x by $-m/l$ and $-q/p$ separately in (1), we see that $-m/l$ and $-q/p$ are roots of the equation

$$ax^2 + bx + c = 0.$$

Also no value of x other than $-m/l$, $-q/p$ can be a root of this equation. In fact, if we suppose that x is replaced by any number other than $-m/l$, $-q/p$, no one of the two factors on the right hand side of (1) will be zero and as such the product will not be zero.

Thus, $-m/l$ and $-q/p$ are the only two roots of the equation, and so the truth set is $\{-m/l, -q/p\}$.

However, if $\quad b^2 - 4ac = 0,$

then we shall have $\quad -m/l = -q/p.$

In fact, as we have already seen,

$$b^2 - 4ac = (lq - mp)^2,$$

so that

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (lq - mp)^2 = 0 \\ &\Rightarrow lq - mp = 0 \\ &\Rightarrow -m/l = -q/p. \end{aligned}$$

The truth set of the equation in this case will be

$$\{-m/l\}.$$

We have thus been able to show the following :

The truth set of

$$ax^2 + bx + c = 0 \quad a, b, c \in \mathbf{Q}, \quad a \neq 0,$$

(*i*) consists of two members *i.e.*, the equation has two roots if $b^2 - 4ac$ is the square of a non-zero rational number ;

(*ii*) consists of only one member if $b^2 - 4ac = 0$;

(*iii*) is void if $b^2 - 4ac$ is not the square of a rational number,

Accordingly, we also say that the equation has

(*i*) two distinct roots,

(*ii*) two equal roots,

(*iii*) no root.

Note. The above method of solving the quadratic equation, no doubt, is very helpful in practice, yet it does not give the roots of the equation in terms of the co-efficients a, b, c. In the following process we shall be able to overcome this difficulty.

Alternative Solution

The process of solving a quadratic equation may as well be exhibited as follows. Of course, it is assumed that $b^2 - 4ac$ is the square of a rational number so that $\sqrt{b^2 - 4ac}$ is meaningful in respect of the set of rational numbers.

We obtain the following chain of equivalent statements.

$$ax^2 + bx + c = 0$$

(Dividing by a, which is non-zero)

$$\Leftrightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$\left(\text{Adding to both sides } -\frac{c}{a}\right)$

$$\Leftrightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

(Adding the square of half of the co-efficient of x to both sides)

$$\Leftrightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\Leftrightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Leftrightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Leftrightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This shows that the required truth set is

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}.$$

In case however, $b^2 - 4ac = 0$, the two members of the truth set become the same so that the truth set contains only one element and is

$$\left\{-\frac{b}{2a}\right\}.$$

Different Procedures of Solving a Quadratic Equation

Let the equation be

$$ax^2 + bx + c = 0 \quad a, b, c \in \mathbf{Q}, a \neq 0$$

Firstly we see if the polynomial

$$ax^2 + bx + c$$

can be factorised by inspection. If we find by inspection that

$$ax^2 + bx + c = (lx + m)(px + q),$$

then the required truth set is

$$\left\{-\frac{m}{l}, -\frac{q}{p}\right\}.$$

Of course in some cases, these two numbers may be the same so that the truth set may consist of only one element.

Suppose now that we are not able to judge the factors by inspection. In this case, *we may go in for factorisation by completing the square* as is shown on page 244.

We could also proceed as on page 252 and obtain the truth set without first knowing the factors.

Finally we could also straightway put down the roots by making substitutions for a, b, c in

$$\left\{\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\right\}.$$

Of course, $b^2 - 4ac$ has to be in all cases the square of a rational number. Different procedures will be indicated in the following examples.

Examples

1. Solve the following quadratic equations.

(*i*) $x^2 - 7x + 12 = 0$ (*ii*) $6x^2 + x - 15 = 0$

(*iii*) $39x^2 - 7x - 22 = 0$ (*iv*) $4x^2 + 12x + 9 = 0.$

(*i*) The product of the co-efficient of x^2 and the constant term is 1×12 *i.e.*, 12. The pair of factors -3, -4 of 12 is such that their sum is $-3 - 4$ which is the co-efficient -7 of x.

We have, therefore,

$$\begin{aligned} x^2 - 7x + 12 &= x^2 - 3x - 4x + 12 \\ &= x(x-3) - 4(x-3) \\ &= (x-3)(x-4). \end{aligned}$$

The truth set of the given equation, therefore, is

$$\{3, 4\}$$

or that 3, 4 are the roots of the equation.

(*ii*) We have

$$\begin{aligned} 6x^2 + x - 15 &= 6\left(x^2 + \frac{1}{6}x - \frac{15}{6}\right) \\ &= 6\left[\left\{x^2 + \frac{1}{6}x + \left(\frac{1}{12}\right)^2\right\} - \left\{\frac{15}{6} + \left(\frac{1}{12}\right)^2\right\}\right] \\ &= 6\left[\left(x + \frac{1}{12}\right) - \frac{361}{144}\right] \\ &= 6\left[\left(x + \frac{1}{12}\right)^2 - \left(\frac{19}{12}\right)^2\right] \\ &= 6\left(x + \frac{1}{12} + \frac{19}{12}\right)\left(x + \frac{1}{12} - \frac{19}{12}\right) \\ &= 6\left(x + \frac{20}{12}\right)\left(x - \frac{18}{12}\right) \\ &= 3\left(x + \frac{5}{3}\right)2\left(x - \frac{3}{2}\right) \\ &= (3x+5)(2x-3). \end{aligned}$$

Thus,

$$6x^2 + x - 15 = (3x+5)(2x-3) \quad \forall\, x \in Q.$$

Now $\qquad 2x - 3 = 0 \Leftrightarrow x = \dfrac{3}{2}$

and $\qquad 3x + 5 = 0 \Leftrightarrow x = -\dfrac{5}{3}.$

The required truth set, therefore, is

$$\left\{\frac{3}{2}, -\frac{5}{3}\right\}.$$

(*iii*) We have

$$a = 39, \quad b = -7, \quad c = -22,$$

so that

$$\begin{aligned} b^2 - 4ac &= (-7)^2 - 4(39)(-22) \\ &= 49 + 4 \times 39 \times 22 \\ &= 49 + 3432 \\ &= 3481 = 59^2. \end{aligned}$$

We know that the roots of the equation

$$ax^2 + bx + c = 0$$

are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Substituting the values of a, b, c we have the roots of the given equation as

$$\frac{7 + 59}{2 \times 39}, \frac{7 - 59}{2 \times 39}$$

i.e.,

$$\frac{11}{13}, \frac{-2}{3}.$$

(*iv*) We have the following chain of equivalent statements.

$$4x^2 + 12x + 9 = 0$$

$$\Leftrightarrow \quad x^2 + 3x + \frac{9}{4} = 0$$

$$\Leftrightarrow \quad x^2 + 3x = -\frac{9}{4}.$$

$$\Leftrightarrow \quad x^2 + 3x + \left(\frac{3}{2}\right)^2 = -\frac{9}{4} + \left(\frac{3}{2}\right)^2$$

$$\Leftrightarrow \quad \left(x + \frac{3}{2}\right)^2 = 0.$$

The only root of the given equation, therefore, is $-\dfrac{3}{2}$.

2. Solve

$$\frac{x-1}{x-2} + \frac{x-3}{x-4} = 3\frac{1}{3}.$$

Solution. The algebraic expressions on the two sides of the equality are meaningful whatever rational number x may be other than 2 and 4. Thus, the domain of x is the set of all rational numbers excluding the numbers 2 and 4.

Multiplying both sides by

$$3(x - 2)(x - 4)$$

which is not zero for all x belonging to the domain referred to above, we see that the given equation is equivalent to

$$3(x - 1)(x - 4) + 3(x - 3)(x - 2) = 10(x - 2)(x - 4)$$

$$\Leftrightarrow 3(x^2 - 5x + 4) + 3(x^2 - 5x + 6) = 10(x^2 - 6x + 8)$$

$$\Leftrightarrow \quad 6x^2 - 30x + 30 = 10x^2 - 60x + 80$$

$$\Leftrightarrow \quad -4x^2 + 30x - 50 = 0$$

$$\Leftrightarrow \quad 2x^2 - 15x + 25 = 0$$

$$\Leftrightarrow \quad (2x - 5)(x - 5) = 0.$$

The required truth set, therefore, is

$$\left\{\frac{5}{2}, 5\right\}.$$

EXERCISES

1. Find the truth sets of the following.

(i) $x^2 - 9 = 0$ (ii) $x^2 - 4x + 4 = 0$

(iii) $4x^2 + 28x + 49 = 0$ (iv) $x^2 + 5x - 84 = 0$

(v) $x^2 + 10x + 16 = 0$ (vi) $8x^2 - 18x + 21 = 0$

(vii) $2y^2 + 8y + 3 = 0$ (viii) $-y^2 + 8y - 15 = 0$

(ix) $3y^2 - y - 2 = 0$ (x) $3y^2 + 11y - 20 = 0$

(xi) $6y^2 + 15y - 77 = 0$ (xii) $4y^2 - 13y + 15 = 0$.

2. Solve the following.

(i) $\dfrac{3x+2}{x-1} + \dfrac{2x+5}{x+2} = 4$ (ii) $\dfrac{3}{x-6} + \dfrac{7}{x-2} = \dfrac{10}{x-4}$

(iii) $\dfrac{(x+1)(x+2)}{(x+4)(x+4)} = \dfrac{x+3}{x+7}$ (iv) $\dfrac{x}{x-2} - \dfrac{x+1}{x-1} = \dfrac{x-3}{x-6} - \dfrac{x-4}{x-7}$.

54. QUADRATIC INEQUALITIES.

We consider the following example to make the procedure of finding the truth sets of quadratic inequalities clear.

Example. Find the truth sets of the following quadratic inequalities.

(i) $x^2 - 3x + 2 > 0$ (ii) $x^2 - 3x + 2 \geqslant 0$

(iii) $x^2 - 3x + 2 < 0$ (iv) $x^2 - 3x + 2 \leqslant 0$.

Solution. (i) We know that the product of two numbers is positive if and only if the numbers are either both positive or both negative. Also

$$x^2 - 3x + 2 = (x-1)(x-2).$$

This gives

$$\{x : x^2 - 3x + 2 > 0\} = \{x : (x-1)(x-2) > 0\}.$$

Again, we have

$$\{x : (x-1)(x-2) > 0\} = \{x : x-1 > 0 \text{ and } x-2 > 0\} \cup \{x : x-1 < 0 \text{ and } x-2 < 0\}.$$

Now, $x - 1 > 0 \Leftrightarrow x > 1$

and $x - 2 > 0 \Leftrightarrow x > 2,$

so that

$$x - 1 > 0 \text{ and } x - 2 > 0 \Leftrightarrow x > 2,$$

Again, $$x - 1 < 0 \Leftrightarrow x < 1$$

and $$x - 2 < 0 \Leftrightarrow x < 2$$

so that

$$x - 1 < 0 \text{ and } x - 2 < 0 \Leftrightarrow x < 1.$$

We see, therefore, that the required truth set is

$$\{x : x > 2\} \cup \{x : x < 1\}$$

so that every rational number less than 1 is a member of the truth set and every rational number greater than 2 is also a member of the truth set. We may as well say that the truth set consists of all rational numbers except 1, 2 and those between 1 and 2.

(*ii*) We see that

$$\begin{aligned}\{x : x^2 - 3x + 2 \geqslant 0\} &= \{x : x^2 - 3x + 2 > 0\} \cup \{x : x^2 - 3x + 2 = 0\}\\ &= \{x : (x - 1)(x - 2) > 0\} \cup \{1, 2\}\\ &= \{x : x > 2\} \cup \{x : x < 1\} \cup \{1, 2\}.\end{aligned}$$

The truth set consists of all rational numbers except those between 1 and 2. We may equivalently say that the truth set consists of all rational numbers less than or equal to 1 and all rational numbers greater than or equal to 2.

(*iii*) We know that the product of two numbers is negative if and only if one of them is positive and the other is negative, so that we have

$$\begin{aligned}\{x : x^2 - 3x + 2 < 0\} &= \{x : (x - 1)(x - 2) < 0\}\\ &= \{x : (x - 1) > 0 \text{ and } (x - 2) < 0\}\\ &\quad \cup \{x : (x - 1) < 0 \text{ and } (x - 2 > 0\}.\end{aligned}$$

Now, $$x - 1 > 0 \Leftrightarrow x > 1$$

and $$x - 2 < 0 \Leftrightarrow x < 2$$

so that

$$x - 1 > 0 \text{ and } x - 2 < 0 \Leftrightarrow x \text{ lies between 1 and 2.}$$

Again, $$x - 1 < 0 \Leftrightarrow x < 1$$

and $$x - 2 > 0 \Leftrightarrow x > 2.$$

But $x < 1$ and $x > 2$ is false.

Therefore,

$$\{x \cdot x - 1 < 0 \text{ and } x - 2 > 0\} = \phi$$

so that

$$\{x : x^2 - 3x + 2 < 0\} = \{x : 1 < x < 2\}.$$

The required truth set is the set of all rational numbers between 1 and 2.

(*iv*) We have

$$\begin{aligned}\{x : x^2 - 3x + 2 \leqslant 0\} &= \{x : (x-1)(x-2) \leqslant 0\} \\ &= \{x : (x-1)(x-2) < 0\} \\ &\qquad \cup \{x : (x-1)(x-2) = 0\} \\ &= \{x : 1 < x < 2\} \cup \{1, 2\} \\ &= \{x : 1 \leqslant x \leqslant 2\}.\end{aligned}$$

Thus, the truth set consists of the rational numbers 1, 2 and all rational numbers between 1 and 2.

EXERCISE

Find the truth sets of the following inequalities.

(*i*) $(1 - x)(x - 2) > 0$

(*ii*) $(x + 1)(x - 3) \leqslant 0$

(*iii*) $(x + 2)(3 - x) > 0$

(*iv*) $(x + 4)(x + 5) \geqslant 0$

(*v*) $(2x - 1)(x - 2) > 0$

(*vi*) $(3x + 2)(4x + 5) < 0$

(*vii*) $x^2 - 9x + 20 \geqslant 0$

(*viii*) $x^2 + x - 20 < 0$

(*ix*) $6x^2 - x - 2 < 0$

(*x*) $-5x^2 - 2x + 3 > 0$

(*xi*) $x^2 - 2x - 8 < 0$

(*xii*) $x^2 - 2x > 15$.

55. PROBLEMS.

1. The sum of the squares of two consecutive odd numbers is 130. Find the numbers.

Solution. Let one of the numbers be $2x - 1$. Then the other will be $2x + 1$. We have, then

$$\begin{aligned}&(2x-1)^2 + (2x+1)^2 = 130 \\ \Leftrightarrow\ &(4x^2 - 4x + 1) + (4x^2 + 4x + 1) = 130 \\ \Leftrightarrow\ &8x^2 - 128 = 0 \\ \Leftrightarrow\ &x^2 - 16 = 0 \\ \Leftrightarrow\ &(x+4)(x-4) = 0.\end{aligned}$$

The truth set of the equation is, therefore,

$$\{4, -4\}.$$

If x is 4, then the numbers are $2 \times 4 - 1, 2 \times 4 + 1$ *i.e.*, 7, 9.

Again if x is -4, then the numbers are $2 \times (-4) - 1, 2 \times (-4) + 1$, *i.e.*, $-9, -7$.

The required numbers are 7, 9 or $-7, -9$.

2. The length of a certain rectangle is 2 metres more than its width. If the length is increased by 6 metres and the width decreased by 2 metres, the area becomes 119 square metres. Find the dimensions of the original rectangle.

Solution. Let x metres be the width of the rectangle.

Its length will be $x + 2$ metres,

The new length and width will be

$$(x + 2) + 6 \text{ metres and } x - 2 \text{ metres.}$$

The area of the new rectangle being 119 square metres, we have

$$\begin{aligned} & (x + 8)(x - 2) = 119 \\ \Leftrightarrow\ & x^2 + 6x - 16 - 119 = 0 \\ \Leftrightarrow\ & x^2 + 5x - 135 = 0 \\ \Leftrightarrow\ & x^2 + 15x - 9x - 135 = 0 \\ \Leftrightarrow\ & x(x + 15) - 9(x + 15) = 0 \\ \Leftrightarrow\ & (x - 9)(x + 15) = 0. \end{aligned}$$

This gives the truth set $\{9, -15\}$.

Although -15 satisfies the equation, we have to reject it as the width of the rectangle cannot be negative.

The width of the rectangle is 9 metres and the length will be 11 metres.

EXERCISES

1. Find the number whose square is 5 less than 6 times the number.

2. The sum of two numbers is 16 and the sum of their squares is 146. Find the numbers.

3. Half of a number added to 3 equals the square of the number. Find the number.

4. The area of a triangle is 12 sq. cm. Its base and height are consecutive even integers. Find the dimensions of the triangle.

5. The perimeter of a rectangle is 44 cm. and its area is 105 sq. cm. Find its dimensions.

6. A rectangular plot is $4\frac{1}{2}$ metres longer than its width. If the area is 90 square metres, find the length of the plot.

SUMMARY

The polynomial

$$ax^2 + bx + c \qquad a, b, c \in \mathbf{Q}, \qquad a \neq 0$$

is expressible as a product of two linear factors if and only if $b^2 - 4ac$ is the square of a rational number.

The equation

$$ax^2 + bx + c = 0 \quad a, b, c \in \mathbf{Q}, a \neq 0.$$

(*i*) has two roots $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\dfrac{-b - \sqrt{b - 4ac}}{2a}$

if $b^2 - 4ac$ is the square of a non-zero rational number.

(*ii*) has one root $\dfrac{-b}{2a}$

if $b^2 - 4ac = 0$.

(*iii*) has no root

if $b^2 - 4ac$ is not the square of a rational number.

REVIEW EXERCISES

1. Which of the following polynomials are expressible as products of linear factors ? Express those, which are thus expressible, as products of linear factors.

(*i*) $x^2 - 5x + 8$ (*ii*) $x^2 + 9x + 18$

(*iii*) $x^2 + 13x + 24$ (*iv*) $10x^2 + 19x - 15$

(*v*) $8x^2 - 29x - 20$ (*vi*) $7x^2 + 18x + 8$.

2. Find the truth sets of the following equations.

(*i*) $x^2 + 12x + 20 = 0$ (*ii*) $3x^2 + x - 10 = 0$

(*iii*) $x^2 + 6x - 27 = 0$ (*iv*) $12x^2 - 20x - 25 = 0$

(*v*) $7x^2 - x = 0$ (*vi*) $2x^2 - 3x + 4 = 0$

(*vii*) $4x^2 - 4x + 1 = 0$ (*viii*) $3x^2 + 5 = 0$

(*ix*) $4x^2 - 25 = 0$

Solve the following.

(*i*) $x + 5 + \dfrac{6}{x-2} = 0$ (*ii*) $3 - x + \dfrac{14}{x} = 0$

(*iii*) $\dfrac{x+2}{x} + \dfrac{3x}{x+4} = 0$ (*iv*) $\dfrac{6}{x^2+9} = \dfrac{1}{x}$.

4. Factorise

(*i*) $x^2 + x - (a^2 - 3a + 2)$ (*ii*) $4x^2 + 12ax + 9a^2 - 8x - 12a$

(*iii*) $4x^2 + 4(3a - 2)x + 9a^2 - 12a$ (*iv*) $2x^2 + 7(1 - a)x - (4a^2 - 8a + 4)$.

5. The length of a rectangle is twice as long as the side of a square The side of the square is 4 cm more than the width of the rectangle. If their areas are equal, find their dimensions.

6. The sum of the base and the height of a triangle is 22 cm. If its area is 52·5 sq. cm., find the base and the height.

7. Find the three consecutive positive integers, the sum of whose squares is 1202.

8. Separate 27 into two positive parts, such that the sum of the squares of the parts is 425.

9. A car covers a distance of 648 km. The number of hours taken for the journey is one half the number representing the speed in km. per hour. Find the time of the journey.

10. In a flight of 600 km. a plane was slowed down by bad weather Its average speed for the trip was reduced by 200 km. per hour and time increased by half hour. What was the actual time of flight ?

APPENDIX

SYSTEMS OF NUMERATION

In chapter 1, it was pointed out that we have different positional schemes of numeration in that we have a positional scheme corresponding to any given natural number greater than one While we have so far confined ourselves to the decimal scheme in respect of which we employ ten symbols

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9,$$

we shall now take up the consideration of schemes of numeration pertaining to any finite number of symbols. This finite number is called the *base* of the scheme.

We start considering the scheme with five symbols

$$0, 1, 2, 3, 4.$$

Let us consider a natural number, which in the decimal scheme is represented as

$$274.$$

We divide 274 by 5, and then divide the quotient thus obtained by 5. We continue this process of dividing the successive quotients by 5 and obtain,

$$\begin{aligned} 274 &= 54 \times 5 + 4 \\ 54 &= 10 \times 5 + 4 \\ 10 &= 2 \times 5 + 0 \\ 2 &= 0 \times 5 + 2. \end{aligned}$$

Thus, we have

$$\begin{aligned} 274 &= (10 \times 5 + 4) \times 5 + 4 \\ &= 10 \times 5^2 + 4 \times 5 + 4 \\ &= (2 \times 5 + 0)\, 5^2 + 4 + 5 + 4 \\ &= 2 \times 5^3 + 0 \times 5^2 + 4 \times 5 + 4. \end{aligned}$$

The expression on the right is written as

$$(2044)_5$$

the subscript 5 referring to the fact that the base used is five.

We have, therefore,

$$(274)_{10} = (2044)_5.$$

Note. 1. It is usual to write a number in the decimal scheme without using the brackets and the subscript 10 so that whenever the base is not specifically mentioned it is supposed to be 10.

2. The reader is reminded here that this process of writing $(2044)_5$ is similar to the one we have for writing 274 in the decimal scheme in place of

$$2 \times 10^2 + 7 \times 10 + 4,$$

where the base is 10.

Conversely, suppose that we have the number

$$(13024)_5.$$

Essentially the number represents

$$1 \times 5^4 + 3 \times 5^3 + 0 \times 5^2 + 2 \times 5 + 4$$

and in the decimal scheme this number is

$$625 + 375 + 10 + 4 \quad i.e., \ 1014,$$

so that we have

$$(13024)_5 = (1014)_{10}.$$

Thus, given a number in the base ten we can convert it to base five and *vice versa*. In an exactly similar fashion we may convert any given number in base ten to a number in any other base and *vice versa*.

For example, in the following we convert

$$(5707)_{10}$$

to the base 7.

We have

$$\begin{aligned} 5707 &= 815 \times 7 + 2 \\ 815 &= 116 \times 7 + 3 \\ 116 &= 16 \times 7 + 4 \\ 16 &= 2 \times 7 + 2 \\ 2 &= 0 \times 7 + 2. \end{aligned}$$

These give successively

$$\begin{aligned} 5707 &= (116 \times 7 + 3)\, 7 + 2 \\ &= 116 \times 7^2 + 3 \times 7 + 2 \\ &= (16 \times 7 + 4) \times 7^2 + 3 \times 7 + 2 \\ &= 16 \times 7^3 + 4 \times 7^2 + 3 \times 7 + 2 \\ &= (2 \times 7 + 2)\, 7^3 + 4 \times 7^2 + 3 \times 7 + 2 \\ &= 2 \times 7^4 + 2 \times 7^3 + 4 \times 7^2 + 3 \times 7 + 2. \end{aligned}$$

Thus, $$(5707)_{10} = (22432)_7.$$

Again, let us convert $(54235)_6$ to base ten. We have

$$\begin{aligned}(54235)_6 &= 5 \times 6^4 + 4 \times 6^3 + 2 \times 6^2 + 3 \times 6 + 5\\ &= 6480 + 864 + 72 + 18 + 5\\ &= 7439\end{aligned}$$

so that $$(54235)_6 = (7439)_{10}.$$

With the help of the process outlined in the above illustrations, given a number expressed in terms of any base, we can obtain the expression for the same in terms of any other base. All that we have to do is that we first convert the given number to the base ten and then from this number in the decimal scheme we go to the number in the required base. We choose to go *via* the decimal scheme as our acquiantance with the same enables us to make computations in terms of this scheme with great ease and convenience. We illustrate this with the help of the following example.

Note. Just as we talked of decimal fractions in chapter 4, we can as well talk of binary fractions, ternary fractions and so on. This, however, is not proposed to be considered here.

Examples

1. Express $(3204)_6$ in the base 7.

Solution. We have

$$\begin{aligned}(3204)_6 &= 3 \times 6^3 + 2 \times 6^2 + 0 \times 6 + 4\\ &= 648 + 72 + 4\\ &= 724.\end{aligned}$$

Again,

$$\begin{aligned}724 &= 103 \times 7 + 3\\ 103 &= 14 \times 7 + 5\\ 14 &= 2 \times 7 + 0\\ 2 &= 0 \times 7 + 2.\end{aligned}$$

Thus, we have

$$724 = (2053)_7$$

so that $$(3204)_6 = (724)_{10} = (2055)_7$$

2. Express $(32)_5$ in terms of base 2.

Solution. We have

$$(32)_5 = 3 \times 5 + 2$$

so that $$(32)_5 = (17)_{10}.$$

Carrying out the computations with the base ten, we have

$$\begin{aligned}17 &= 8 \times 2 + 1\\ 8 &= 4 \times 2 + 0\\ 4 &= 2 \times 2 + 0\\ 2 &= 1 \times 2 + 0\\ 1 &= 0 \times 2 + 1.\end{aligned}$$

These give

$$17 = 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2 + 1$$

so that

$$(32)_5 = (17)_{10} = (10001)_2.$$

In case the base is more than ten, we have to have symbols other than the ten symbols

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

Suppose we wish to express a number in the base twelve, then we need twelve symbols In addition to the ten symbols for the decimal scheme, we need two more symbols. We denote these by α and β. Thus, we have the twelve symbols

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \alpha, \beta.$$

Here the symbols α and β stand for the numbers ten and eleven respectively.

Example

Exhibit $(51778)_{10}$ in terms of the base twelve.

Solution. Carrying out successive divisions by 12 in terms of the decimal scheme, we have

$$\begin{aligned} 51778 &= 4314 \times 12 + 10 \\ 4314 &= 359 \times 12 + 6 \\ 359 &= 29 \times 12 + 11 \\ 29 &= 2 \times 12 + 5 \\ 2 &= 0 \times 12 + 2. \end{aligned}$$

Thus, we get

$$(51778)_{10} = (25\,\beta\,6\,\alpha)_{12}.$$

EXERCISES

1. Convert the following to base ten :

(*i*) $(1111011)_2$ (*ii*) $(210221)_3$
(*iii*) $(20130)_4$ (*iv*) $(40213)_5$
(*v*) $(53400)_6$ (*vi*) $(6666)_7$
(*vii*) $(732104)_8$ (*viii*) $(\alpha\,00\,\alpha\,2)_{11}$
(*ix*) $(3\,\alpha\,0\,\beta.\alpha\,5)_{12}$.

2. Exhibit the following as indicated :

(*i*) $(45)_{10}$ in base 2 (*ii*) $(725)_{10}$ in base 6
(*iii*) $(95)_{10}$ in base 3 (*iv*) $(213)_{10}$ in base 4.
(*v*) $(2345)_{10}$ in base 5 (*vi*) $(77335)_{10}$ in base 7
(*vii*) $(4444)_{10}$ in base 8 (*viii*) $(553370)_{10}$ in base 9
(*ix*) $(1000001)_{10}$ in base 11 (*x*) $(730245)_{10}$ in base 12.

3. Express the following as indicated :

(*i*) $(\beta\alpha)_{12}$ in base 2 (*ii*) $(\alpha\,0)_{12}$ in base 5
(*iii*) $(1\,\alpha)_{11}$ in base 6 (*iv*) $(101010)_2$ in base 7

(v) $(2341)_7$ in base 9 (vi) $(34254)_6$ in base 12

(vii) $(331100)_9$ in base 11 (viii) $(2010)_3$ in base 2

(ix) $(43205)_8$ in base 7 (x) $(3021)_4$ in base 8.

BINARY SYSTEM

In view of the great importance the Binary System of numeration has achieved recently in computer technology, we exhibit in the following, how addition, multiplication and subtraction can be carried out in this scheme. Before working out examples, we give below the addition and multiplication tables in this scheme. The base is two and the symbols involved are 0 and 1.

Addition Table

+	0	1
0	0	1
1	1	10

Multiplication Table

×	0	1
0	0	0
1	0	1

As we are working with base 2 throughout, we agree to omit the subscript 2.

Examples

1. Find the sum of 1101 and 1110.

Solution.

$$\begin{array}{r} 1\,1\,0\,1 \\ +\ \ 1\,1\,1\,0 \\ \hline 1\,1\,0\,1\,1 \end{array}$$

From the addition table, we have the sum of 1 and 0 is 1 and that of 0 and 1 is also 1. We put below the line 1 and 1, the sum of the first and second columns on the right. Also the sum of 1 and 1 is 10. Below the third column we put 0 and carry 1 to the fourth column. Again the sum of this 1 carried over and 1 in the fourth column will be the same as the sum of 1 and 10 which is 11.

Thus, we have

$$1101 + 1110 = 11011.$$

2. Find the product of 1010 and 101.

Solution.

$$\begin{array}{r} 1\,0\,1\,0 \\ \times \qquad 1\,0\,1 \\ \hline 1\,0\,1\,0 \\ 0\,0\,0\,0\ \ \\ 1\,0\,1\,0\ \ \ \ \\ \hline 1\,1\,0\,0\,1\,0 \end{array}$$

The multiplication is computed in a manner similar to the one we adopt in the decimal scheme, by using the multiplication table. However, we have to use the addition table as well. For example, while adding the fourth column $1 + 0 + 1$ we have the sum 10. We put this zero below the column and carry 1 to the next column Thus, we have

$$1010 \times 101 = 110010.$$

3. Subtract 101 from 11010.

Solution. We write the first number below the second.

$$\begin{array}{r} 1\,1\,0\,1\,0 \\ -\quad 1\,0\,1 \\ \hline 1\,0\,1\,0\,1. \end{array}$$

In the column on the extreme right we see that $1 > 0$, so that we cannot subtract 1 from 0. We borrow 1 from the second place in the first row. It becomes 10 in the first place. Subtracting 1 from 10 we get 1 which we put below the line. We are left with 0 in the second place, subtracting 0 from which we get 0 below the line in the second place. Again from the fourth place we borrow 1 which becomes 10 in the third place. Subtracting 1 from it we get 1 in the third place below the line. The remaining places below the line will obviously have 0 and 1 respectively. Thus, we have

$$11010 - 101 = 10101.$$

The reader may check that 101 added to 10101 gives 11010.

EXERCISES

1. Compute the following, the base being two.

(*i*) $1111 + 1011$ (*ii*) $100100 + 11011$
(*iii*) 1110×1001 (*iv*) 1010×1010
(*v*) $101101 - 10011$ (*vi*) $10010 - 1001$.

2. Write the following numbers in ascending order.

(*i*) 1000, 1010, 111, 1100
(*ii*) 101, 11, 111, 100, 110
(*iii*) 1010, 1001, 1100, 1000.

3 Replace ? by the appropriate symbol $>$ or $<$.

(*i*) 1010 ? 1001 (*ii*) 100101 ? 101101
(*iii*) 111 ? 1000 (*iv*) 10110 ? 10011.

Test Papers

Test Paper I

1. (*a*) Given that

$$A = \{2, 0, 3, 7, 4, 8\}, \qquad B = \{7, 9, 6, 8, 0, 11\},$$

list the sets $A \cup B$, $A \cap B$.

(*b*) Give five rational numbers which are not integers and five integers which are not natural numbers.

2. Definite the HCF of two given numbers.

Put down the sets of factors of 24 and 42. Are these sets finite or infinite? Find their intersection set. Put down the least and the greatest members of this set. Also find the HCF of the two numbers.

3. What do you mean by the statement 'a divides b' where a and b are natural numbers ?

Show that, if a divides b and b divides c, then a divides c.

Show that $$a \mid b \Rightarrow a \leqslant b.$$

4. (*a*) Define a prime number and show that the set of primes is infinite.

(*b*) Express the numbers 27648 and 3600 as products of prime factors and find their LCM.

5. (*a*) Find the truth set of

$$3x + 2 = 1,$$

the domain of the variable being **N**. How does the truth set change if the domain of the variable is taken as **Q** ?

(*b*) Find the truth set of

$$5x + 3 \leqslant 28,$$

the domain of the variable being the set of *odd* natural numbers.

6. (*a*) State the distributive law in the set **Q** of rational numbers. Assuming this and the commutative and associative laws of addition and multiplication, show that

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \forall\, a, b \in \mathbf{Q}.$$

(*b*) For rational numbers a, b, c, show that

$$a > b \Leftrightarrow a + c > b + c.$$

7. (*a*) When is a fraction called a decimal fraction ? Show that the sum and the product of two decimal fractions is a decimal fraction.

(*b*) Arrange the following sets of numbers in ascending order.

(*i*) $\left\{\frac{2}{3}, \frac{13}{15}, 0, -{\cdot}75, 1{\cdot}25\right\}$

(*ii*) $\{-1{\cdot}27, -3{\cdot}24, -2{\cdot}5, {\cdot}04, {\cdot}75\}$.

8. (*a*) Define that absolute value $|x|$ of a rational number x. Find the truth set of

$$|2x - 3| = 4,$$

the domain of x being **Q**.

(*b*) Find the truth set of

$$\begin{cases} 7x - 5y + 11 = 0 \\ 2x + 3y - 7 = 0. \end{cases}$$

9. (*a*) Find the condition that the equations

$$ax + b = 0$$

and

$$lx = m$$

are consistent.

(*b*) Express, if possible,

$$8x^2 + 13x - 6$$

as a product of linear factors with rational co-efficients.

10. (*a*) Find the truth set of

$$6x^2 - 43x + 20 = 0, x \in \mathbf{Q}.$$

(*b*) There are two examination rooms A and B. If 10 candidates are sent from A to B, the number in each is the same while if 20 are sent from B to A the number in A is double the number in B. Find the number of candidates in each room.

Test Paper II

1. (*a*) Given that

$$A = \{1, 3, 5, 7\}, B = \{2, 4, 6, 8\}, C = \{0, 4, 5\},$$

what are the sets $A \cap B$, $B \cap C$, $(A \cup B) \cap C$?

(*b*) Give five fractions which are not integers and five integers which are not fractions.

2. Define the LCM of two given numbers.

Put down the sets of multiples of 4 and 6. Are these sets finite or infinite? Find their intersection set. Does this set have the greatest member? What is the least member of this intersection set? Find the LCM of the two numbers.

3. What do you mean by the statement 'a is a factor of b' where a and b are natural numbers?

Show that if a is a factor of b as also of c, then a is a factor of $b + c$.

Show that $$a \mathrm{I} b, b \mathrm{I} a \Rightarrow a = b.$$

4. (*a*) State and prove the Unique Prime Factorisation Theorem.

(*b*) Show that a product of three consecutive natural numbers is divisible by 6.

5. (*a*) Find the truth set of

$$2x = 6,$$

the domain of the variable being **Q**. How does the truth set change if the domain of the variable is taken as **N** or **F** or the set of even natural numbers?

(*b*) Find the truth set of

$$2x + 3y = 11$$

the domain of the variables being **N**. Is this truth set finite or infinite? Will the truth set be finite if the domain is changed to **Q**?

6. (*a*) For rational numbers a, b, c show that

$$a = b \Leftrightarrow a + c = b + c.$$

(*b*) For any rational numbers x and y, show that

$$-(x + y) = (-x) + (-y).$$

7. (*a*) Define a decimal fraction.

If a / b and c / d are two decimal fractions, is $a / b \div c / d$ always a decimal fraction? Give reasons in support of your answer.

(*b*) Arrange the following sets of numbers in descending order.

(*i*) $\left\{\frac{1}{2}, \frac{3}{4}, 0, -\frac{1}{4}, -\frac{5}{6}\right\}$

(*ii*) $\{0, \cdot 74, \cdot 77, -\cdot 73, -\cdot 79\}$.

8. (*a*) Show that the quadratic equation

$$ax^2 + bx + c = 0 \qquad a, b, c \in \mathbf{Q}, a \neq 0$$

has the only root $-b / a$ if $b^2 - 4ac = 0$.

(*b*) Express, if possible,

$$10x^2 + 15x - 4$$

as a product of linear factors with rational co-efficients.

9. (*a*) If a, b are different rational numbers, find the condition that the equations

$$\begin{aligned} x + y + 1 &= 0 \\ ax + by + c &= 0 \\ a^2x + b^2y + c^2 &= 0 \end{aligned}$$

are consistent, c being a member of **Q**.

(*b*) Find the truth set of the following system:

$$\begin{cases} 2x - 3y + z - 5 = 0 \\ 3x + 4y - 5z + 8 = 0 \\ x + 24y - 19z + 40 = 0. \end{cases}$$

10. (*a*) Find a fraction which is such that when 15 is added to its numerator it becomes 3 and when 11 is added to its denominator it becomes 1/3.

(*b*) A is 2 years older than B, B is three years older than C and the sum of their ages is half the age of D. In 10 years' time the sum of the ages of A, B, C will be equal to the then age of D. Find their present ages.

Test Paper III

1. (*a*) Given that

$$A = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}, B = \left\{1, 2, 3, 4\right\}, C = \left\{\frac{1}{2}, 3\right\}$$

what are the sets

$$A \cap B, B \cup C, (A \cup B) \cup C, A \cap (B \cup C)?$$

(*b*) Give five fractions which are not natural numbers. Give, if possible, a natural number which is not a fraction.

2. Define the HCF of two given numbers.

Put down the sets of factors of 63, 45, 27. Find the intersection set of these. What are the greatest and least members of this set? Find the HCF of the three numbers.

3. What do you mean by saying that the relation 'is a factor of' is reflexive in the set of natural numbers?

Show that every natural number other than 1 has at least two factors. What are the factors of the number 1?

4. (*a*) State and prove Gauss's Theorem.

(*b*) Show that every natural number other than 1 admits of a prime factor.

5. (*a*) Given any two rational numbers x and y, is

$$x - y$$

always meaningful ? Justify your answer. How would your answer change if x and y were assumed to be any fractions ?

(*b*) Assuming the various *field* properties of Q, show that

$$(-x)(y) = -(xy) \quad \forall\, x, y \in Q$$

and
$$(-x)(-y) = xy \qquad \forall\, x, y \in Q.$$

6. (*a*) Find the truth set of

$$2x + 3y + 2z = 11,$$

the domain of the variables being N.

(*b*) Show that

$$a > b,\, c > 0 \Rightarrow ac > bc$$

where a, b, c are members of Q.

7. (*a*) What do you mean by the statement "The set of fractions is order dense" ? Prove it.

(*b*) Which of the following statements are true ?

(*i*) $-7 > 3$ (*ii*) $-2 > 3$ (*iii*) $\frac{3}{4} < \frac{13}{14}$.

8. (*a*) Simplify

$$\frac{3x^2y + 8xy^2}{2x + 3y} \cdot \frac{4x^2 + 12xy + 9y^2}{3xy^2 + 8y^3}.$$

(*b*) Find the truth set of

$$4x^2 + 3x - 10 \leqslant 0$$

the domain of x being Q.

9. (*a*) Find the condition that the quadratic equation

$$ax^2 + bx + c = 0 \qquad a, b, c \in Q, \qquad a \neq 0$$

has two roots. Find the same.

(*b*) Show that the following equations are dependent.

$$x + 7y - 3z + 2 = 0$$
$$4x - 2y + z - 3 = 0$$
$$3x - 39y + 17z - 16 = 0.$$

10. (*a*) A boat goes upstream 15 km. and downstream 22 km. in 5 hours. It also goes upstream 20 km. and downstream $27\frac{1}{2}$ km. in $6\frac{1}{2}$ hours. Find the speed of the stream and that of the boat in still water.

(*b*) The product of two natural numbers is 45. If one is 4 less than the other, find the two numbers.

Test Paper IV

1. (*a*) Given that

$$A = \{0, -\tfrac{2}{3}, -\tfrac{1}{2}, -1\}, B = \{1, \tfrac{1}{2}, \tfrac{2}{3}\}, C = \{0, \tfrac{1}{2}, -\tfrac{1}{2}\},$$

which of the following statements are true ?

$$C \subset B, B \subset C, C \subset (A \cup B), C \subset (A \cap B).$$

(*b*) Give four rational numbers which are not fractions. Give if possible a fraction which is not a rational number.

2. Define the LCM of three given numbers.

Put down the sets of multiples of 5, 10 and 15. Find their intersection set. Does this set have a greatest member ? Find the LCM of the three numbers.

3. (*a*) Show that 'a is a multiple of b' implies that a is greater than or equal to b.

(*b*) When are two numbers said to be co-prime ? Put down five pairs of co-primes.

4. (*a*) If h is the HCF of a and b, show that the HCF of ma and mb is mh.

(*b*) Show that the product of two numbers is equal to the product of their HCF and LCM.

5. State the various basic *field* properties of **Q** and deduce the following results :

$$(-x)(y) = -(xy)$$

$$(-x)(-y) = xy,$$

and

$$(x - y)^2 = x^2 - 2xy + y^2.$$

6. (*a*) Show that

$$c \neq 0, ac = bc \Rightarrow a = b$$

where

$$a, b, c \in \mathbf{Q}.$$

(*b*) Assuming the commutative, associative laws of addition of fractions, show that

$$(x + y) + (z + u) = (x + u) + (z + y) \qquad \forall\ x, y, z, u \in \mathbf{F}.$$

Give reasons for each step.

7. (*a*) Show that between any two fractions there is always another fraction.

Will the statement hold if we were dealing with natural numbers rather than fractions ?

(*b*) (*i*) Give five fractions beween $\frac{1}{2}$ and $\frac{3}{4}$.

(*ii*) Give all the natural numbers between 3 and 7.

8. (*a*) Solve the equation

$$\frac{9x+8}{18}+\frac{34}{27}+\frac{4x+}{9} = \frac{21x-8}{18};\ x \in Q.$$

(*b*) Is it possible that the equation

$$ax^2+bx+c=0 \qquad a, b, c \in Q, \qquad a \neq 0,$$

has more than two roots ? Justify your answer. Also find the condition that the equation has no root.

9. (*a*) Find the truth set of the following system.

$$\begin{cases} \dfrac{2}{x}+\dfrac{1}{y}-\dfrac{5}{z}+11 = 0 \\ \dfrac{1}{x}-\dfrac{3}{y}+\dfrac{2}{z}-1 = 0 \\ -\dfrac{7}{x}+\dfrac{2}{y}-\dfrac{4}{z}+15 = 0 \end{cases}$$

(*b*) Show that the following equations are consistent.

$$x+y-4=0$$
$$3x-2y+3=0$$
$$-4x+7y-17=0$$

10. (*a*) Find the truth set of

$$(3-x)(2x+1) > 0;\ x \in Q.$$

(*b*) A man wishes to invest Rs. 31,500 in two stocks 15% at 143 and $10\frac{1}{2}$% at 91 so as to have the same income from each. How much should he invest in each stock ?

Test Paper V

1. (*a*) Given that

$$A=\left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}, \quad B=\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4\right\}$$

what are the sets $A \cup B$, $A \cup B$?

Which of the following statements are true

$$A = B, A \subset B, B \subset A\ ?$$

(*b*) Give seven rational numbers which are not natural numbers. Does there exist a natural number which is not a rational number ?

2. Define the HCF of three given numbers.

Put down the sets of factors of 45, 63, 20. Find their intersection set. Is it finite or infinite ? What are its greatest and least members ? Find the HCF of the numbers.

3. (*a*) Given two natural numbers a and b such that $a > b$ and b is not a factor of a, show that there exist unique natural numbers q and r such that

$$a = bq + r \qquad r < b.$$

(*b*) Distinguish between the concepts of

(*i*) a prime number and (*ii*) a pair of co-primes.

4. (*a*) Show that h is the HCF of two numbers a and b if and only if h is a factor of a and b and the two numbers a/h and b/h are co-primes.

(*b*) Express the numbers 375, 240, 390, 585 as products of prime factors and find their HCF.

5. (*a*) Assuming the field properties of **Q**, show that

$$\frac{1}{xy} = \frac{1}{x} \cdot \frac{1}{y} \qquad \forall\, x, y \in \mathbf{Q}_0.$$

(*b*) Show that

$$x > y,\ z < 0 \Rightarrow xz < yz, \text{ where } x, y, z \in \mathbf{Q}.$$

6. (*a*) If the expression $\dfrac{3x + 4}{5 - 2x}$ is meaningful where $x \in \mathbf{F}$, what restrictions have to be imposed on x ?

(*b*) Find the truth set of

$$3 + 8x \geqslant 11,$$

given that $x \in \mathbf{N}$. Is this truth set finite or infinite ?

7. (*a*) Show that between two different rationals, there will always be a rational number.

Will the result hold if we were dealing with integers rather than rationals ? Give a counter example to show that this is not so.

(*b*) What is the set of integers between -11 and -12 ?

8. (*a*) Solve for x

$$\frac{x + 3}{2x - 5} - \frac{3x - 4}{6x + 11} = 0\,;\ x \in \mathbf{Q}.$$

(*b*) Find the truth set of

$$|\,2x + 7\,| \leqslant 13\,;\ x \in \mathbf{Q}.$$

9. (*a*) Show that the equations

$$2x - y + 7z + 5 = 0$$
$$6x - 3y + 21z + 7 = 0$$

are inconsistent.

(*b*) Solve the following system of equations:

$$\frac{2}{x} - \frac{3}{y} + 5 = 0$$

$$\frac{3}{x} + \frac{2}{y} + 4 = 0.$$

10. (*a*) If three taps are opened together, a cistern is filled in $7\frac{1}{2}$ hours. One of the taps can fill it in 6 hours and another in 15 hours. How does the third tap work ?

(*b*) Anil goes a distance of 30 km. on his bicycle. The number of hours taken by him is one less than his average speed in km. per hour. Find the time taken by him to complete the journey.

Answers

CHAPTER 1

Page 3.

(*iii*) , (*v*) , (*vii*) , (*viii*) , (*ix*) : Yes.

(*i*) , (*ii*) , (*iv*) , (*vi*) : No.

Page 5.

1. Commutative Property.
3. (*i*) 12, (*ii*) 15, (*iii*) 19, (*iv*) 9, (*v*) 39, (*vi*) 105, (*vii*) 33, (*viii*) 14.

Page 7.

1. Associative Property.
3. (*i*) 5, (*ii*) 13, (*iii*) 11, (*iv*) 12, (*v*) 13, (*vi*) 17.

Page 8.

1. (*i*) 58, (*ii*) 400, (*iii*) 531, (*iv*) 733, (*v*) 124, (*vi*) 228, (*vii*) 58, (*viii*) 177.
2. (*i*) 8, (*ii*) 4, (*iii*) 14, (*iv*) 7, (*v*) 9, (*vi*) 9.
3. (*i*) 30, (*ii*) 92, (*iii*) 209, (*iv*) 209.

Page 9.

1. Commutative Property.
3. (*i*) 24, (*ii*) 69, (*iii*) 47, (*iv*) 61.

Page 10.

1. Associative Property.
3. (*i*) 25, (*ii*) 21, (*iii*) 9, (*iv*) 13, (*v*) 7, (*vi*) 111

Page 11.

1. (*i*) 27, (*ii*) 27, (*iii*) 5, (*iv*) 15, (*v*) 13, (*vi*) 4.
2. (*i*) 380, (*ii*) 577300, (*iii*) 8400, (*iv*) 1624000, (*v*) 360, (*vi*) 390, (*vii*) 2460, (*viii*) 1650.

Page 13.

1. Distributive Property
3. (*i*) 13, (*ii*) 23, (*iii*) 27, (*iv*) 23, (*v*) 41, (*vi*) 9.

4. (*i*) 9, *CA*, *AA*, (*ii*) 7, *D*, *CM*, *CA*, (*iii*) 27, *CA*, (*iv*) 109, *AA*, (*v*) 4, *D*, (*vi*) 15, *D*, *CM*, *CA*, (*vii*) 22, *CM*, *D*, (*viii*) 72, *D*, *CA*, *CM*, *AM*, (*ix*) 12, *D*, *AM*.

Page 14.

1. (*i*) 2900, (*ii*) 3700, (*iii*) 2960, (*iv*) 860, (*v*) 390, (*vi*) 820.
2. (*i*) 137, (*ii*) 77, (*iii*) 6700, (*iv*) 37000, (*v*) 9400, (*vi*) 13700, (*vii*) 8900, (*viii*) 188, (*ix*) 1740, (*x*) 999000, (*xi*) 99990000, (*xii*) 8370000,

Page 15.

(*i*) 125, (*ii*) 36, (*iii*) 256, (*iv*) 343, (*v*) 625, (*vi*) 81, (*vii*) 10000, (*viii*) 25, (*ix*) 729, (*x*) 256, (*xi*) 3200000, (*xii*) 216.

Page 15.

1. (*i*) 7^6, (*ii*) x^6, (*iii*) x^4, (*iv*) x^3y, (*v*) x^4y^6, (*vi*) x^3y^4, (*vii*) x^9y^9, (*viii*) 5^6, (*ix*) 3^6, (*x*) 7^7, (*xi*) 5^7, (*xii*) 8^4.
2. (*i*) 36, (*ii*) 108, (*iii*) 2000, (*iv*) 4000,

Page 16.

(*i*) $12a$ (*ii*) $6a$ (*iii*) $6a + 3b$
(*iv*) $4a + 7b$ (*v*) $3a + 8b$ (*vi*) $2x + y + z$.

Page 17.

(*i*) 2 (*ii*) 2 (*iii*) 9
(*iv*) 4 (*v*) 2 (*vi*) 4
(*vii*) 5 (*viii*) 6 (*ix*) 4
(*x*) 6.

Pages 17-18.

1. (*i*) 1 (*ii*) 2 (*iv*) 3 (*vi*) 4
(*vii*) 6 (*ix*) 7 (*x*) 11.
2. (*i*) 2 (*iv*) 3.
3. (*i*) 2 (*iv*) 3 (*v*) 5.
4. (*i*) a (*ii*) ab (*iii*) a^2b^2 (*iv*) b
(*v*) a^2b (*vi*) $2a$ (*vii*) a^2 (*viii*) $4a^2b^2$
(*ix*) $9b^4c^6$.

Page 21.

(*i*) $3(x + y)$ (*ii*) $a(x + y)$ (*iii*) $2x(y + z)$
(*iv*) $3x(y + 2z)$ (*v*) $x(4y + 5z)$ (*vi*) $(a + b)(2x + 3y)$
(*vii*) $2y(4 + 3x)$ (*viii*) $a(a + 2)$ (*ix*) $x^2(x + 1)$
(*x*) $xy(3x + 5y)$ (*xi*) $3(x + y + z)$ (*xii*) $3(ax + 2by + 3c)$
(*xiii*) $x(yz + xy + yz)$ (*xiv*) $8(x + y)$ (*xv*) $(2 + 3a)(x + 3y)$
(*xvi*) $(a + b)(2x + 5y)$ (*xvii*) $(a + b)(x + y)$
(*xviii*) $(2x + 3)(3y + 4)$ (*xix*) $(4a + 1)(b + 4)$
(*xx*) $(x + 1)(y + 1)$ (*xxi*) $(u + 2)(v + 2)$ (*xxii*) $(a + y)(a + x)$.

Page 23.

2. (*i*) $12 > 11$ (*ii*) $15 > 5$ (*iii*) $24 > 21$ (*iv*) $37 > 12$.
3. True : (*ii*), (*v*), (*vii*), (*x*), (*xii*), (*xiv*), (*xvi*), (*xix*).
   False : (*i*), (*iii*), (*iv*), (*vi*), (*viii*), (*ix*), (*xi*), (*xiii*), (*xv*), (*xvii*), (*xviii*), (*xx*).

Pages 34-35.

1. (*i*) {18} (*ii*) {12} (*iii*) {26}
   (*iv*) {3} (*v*) {26} (*vi*) {57}
   (*vii*) empty (*viii*) {8} (*ix*) empty
   (*x*) {3} (*xi*) {4} (*xii*) {2}
   (*xiii*) empty (*xiv*) {7} (*xv*) empty.
2. (*i*) {3 , 4 , 5 , ...} (*ii*) {4 , 5 , 6 , ...} (*iii*) {6 , 7 , 8 , ...}
   (*iv*) {1 , 2 , 3 , ...} (*v*) empty (*vi*) { 1 }
   (*vii*) empty, (*viii*) {2 , 3 , 4 , ...} (*ix*) {1 , 2 , 3 , ...}.
3. (*i*) { 3} (*ii*) empty (*iii*) { 1 }
   (*iv*) {1 , 3} (*v*) {5 , 7 , 9 , ...} (*vi*) {1 , 3 , 5 , 7}.

Page 35.

(*i*) (1 , 15) , (2 , 13) , (3 , 11) , (4 , 9) , (6 , 5) , (7 , 3) , (8 , 1)
(*ii*) (3 , 5) , (6 , 3) , (9 , 1) (*iii*) no solution (*iv*) no solution.

Page 38.

1. (*i*) 14 (*ii*) 2.
2. (*i*) 1 , 2 , 3 (*ii*) 10 , 11 , 12 , ... (*iii*) 3.
3. (*i*) 104 (*ii*) 1 (*iii*) 11 , 12 , 13 , ...
   (*iv*) no solution (*v*) {1 , 2} (*vi*) {1 , 2}.

Page 40.

1. (*i*) 2 (*ii*) 1.
2. (*i*) multiples of 3. (*ii*) 1 , 3 (*iii*) all odd numbers.
   (*iv*) 2 , 5 , 11 , 29 (*v*) 1 , 3 , 4 , 5 , 6 (*vi*) 9 , 16 , 23 , 30 , ...
3. (*i*) {14} (*ii*) { 2 } (*iii*) {16}
   (*iv*) {15 , 18 , 21 , ...} (*v*) {1 , ? , 4} (*vi*) { 1 }.

Page 44.

(*iii*) {1 , 2 , 3 , ...} (*iv*) {1 , 2 , 3 , 4} (*ix*) { 1 }
(*x*) {3 , 4 , 5 , ...}.

Page 44.

True : (*i*).
False · (*ii*) , (*iii*) , (*iv*).

Pages 44-45.

True : (*iii*) , (*iv*) , (*v*) , (*vii*)
False : (*i*) , (*ii*) , (*vi*) , (*viii*)

Page 45.

(*i*) {1 , 13} (*ii*) {1 , 5 , 25} (*iii*) {1 , 2 , 3 , 4 , 6 , 9 , 12 , 18 , 36}
(*iv*) {1 , 53} (*v*) { 1 } (*vi*) {1 , 2 , 3 , 4 , 6, 8 , 12 , 24}
(*vii*) {1 , 2 , 3 , 4 , 5 , 6 , 10 , 12 , 15 , 20 , 30 , 60}
(*viii*) {1 , 3 , 5 , 9 , 15 , 45}.

Page 47.

(*i*) ϕ , { 1 } , { 3 } , { 5 } , {1 , 3} , {3 , 5} , {1 , 5} , {1 , 3 , 5}
(*ii*) ϕ , { 7 } (*iii*) ϕ , { 2 } , { 6 } , {2 , 6} (*iv*) ϕ.

Page 48.

(*i*) 2, does not exist (*ii*) 2 , 256 (*iii*) 10, does not exist
(*iv*) 10 , 70 (*v*) 1 , 1.

Page 48.

1. (*i*) {1 , 2 , 3 , 4} (*ii*) {1 , 3 , 4 , 6 , 7 , 9}
(*iii*) {1 , 2 , 3 , 4 , 5 , 6 , 12 , 15}.
{1 , 2 , 4 , 6 , 7 , 8 , 9}.
4. Commutativity and Associativity.

Page 49.

1. (*i*) {1 , 2 , 3} (*ii*) ϕ (*iii*) {1 , 3}.
2. ϕ.

Page 52.

(*i*) 5 , 2 (*ii*) 286 is divisible by 11 (*iii*) 33 , 14
(*iv*) 413 , 1 (*v*) 222 , 5 (*vi*) 100 , 7.

Page 54.

3. (*i*) $x(x+1) = 18$, smaller one being x.
(*ii*) $x + (x + 13) = 57$, ,, ,, ,, .
(*iii*) $x + (x + 13) = 57$, x denoting the number of girls.
(*iv*) $x + (2x + 3) = 63$, x denoting the age of the son.
(*v*) $2\{x + (x + 3)\} = 42$, x denoting the width of rectangle.

Pages 56-58.

(1) 28, 29 (2) No such numbers exist. (3) 33, 35 (4) 80, 88 (5) 26, 27, 28 (6) 34, 36, 38 (7) 107, 109, 111 (8) 29, 75 (9) 2 (10) 8 (11) {1, 2, 3, 4, 5, 6} (12) {2, 3, 4, 5, 6, 7} (13) No (14) No (15) 30 Km. (16) {1, 2, 3, 4, 5, 6} (17) {4, 5} (18) 3 (19) 6, 11, 19 (20) 59, 60, 61 (21) 51, 41 (22) 18 (23) Krishan 56, Shyam 112, Ram 132 (24) 11, 39 (25) 35, 10 (26) Son 20, Father 45 (27) Son 21, Father 39 (28) 53 (29) 42.

REVIEW EXERCISES : PAGES 58-60

1. {2, 3, 5, 7, 8, 9, 13, 17}, {4, 5, 6, 7, 8, 9, 12, 14, 16, 17}
{2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 16}
{7, 8, 9}, ϕ , ϕ
{2, 7, 3, 9, 8, 13}, {7, 8, 9}
{2, 3, 7, 8, 9, 13}, {7, 8, 9}.
2. {1, 3, 9}, {1, 3, 9}, {1, 3}, {1, 3}.
3. {x : x is a multiple of 24},
4. (*i*) {1, 2, 3} (*ii*) {5, 6, 7} (*iii*) $\{x : x \geqslant 4\}$.
5. (*i*) {1, 2, 4} (*ii*) {x : x is a multiple of 15}
(*iii*) {13, 19, 25,...}.
8. $2^{10} > 1000$.
11. (9, 4), (8, 5), (8, 4), (8, 3), (7, 6), (7, 5), (7, 4), (7, 3), (7, 2), (6, 5), (6, 4) (6, 3), (6, 2), (6, 1), (5, 4), (5, 3), (5, 2), (5, 1), (4, 3), (4, 2) (4, 1), (3, 2), (3, 1), (2, 1).
15 (*i*) 22 (*ii*) 51 (*iii*) 13 (*iv*) 3 (*v*) 14 (*vi*) no solution (*vii*) no solution (*viii*) 6 (*ix*) no solution (*x*) no solution (*xi*) no solution (*xii*) 3 (*xiii*) no solution (*xiv*) 5 (*xv*) 2.
16. (*i*) $\{x : x > 11\}$ (*ii*) $\{x : x < 14\}$ (*iii*) ϕ (*iv*) ϕ (*v*) $\{x : x < 12\}$
(*vi*) $\{x : x < 7\}$ (*vii*) {1, 2, 3} (*viii*) $\{x : x > 2\}$ (*ix*) $\{x : x \geqslant 6\}$ (*x*) N
(*xi*) $\{x : x \geqslant 2\}$ (*xii*) {1, 2}
(*xiii*) {70, 77, 84, 91,...} (*xiv*) {3, 6, 9, 12, 15, 18, 21}
(*xv*) {3, 6, 9, 12, 15} (*xvi*) {3} (*xvii*) {1, 3}
(*xviii*) $\{x : x \geqslant 8\}$ (*xix*) {1, 2, 3} (*xx*) $\{x : x \geqslant 4\}$
(*xxi*) ϕ (*xxii*) N (*xxiii*) $\{x : x \neq 7\}$
(*xxiv*) $\{x : x \geqslant 6\}$ (*xxv*) {1, 2, 3, 4, 5, 6}
(*xxvi*) ϕ (*xxvii*) $\{x : x \geqslant 6\}$
(*xxviii*) {1, 2}.
17. (*i*) (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1).
(*ii*) ϕ (*iii*) $\{(3x + 4, x) : x \in N\}$
(*iv*) $\{(3x + 3, x) : x \in N\}$ (*v*) ϕ
(*vi*) $\{(x, y) : y \geqslant x + 3\}$ (*vii*) $\{(x, y) : y \geqslant x + 3\}$
(*viii*) (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (6, 1), (6, 2), (6, 3), (6, 4), (7, 1), (7, 2), (7, 3), (8, 1), (8, 2), (8, 3), (9, 1), (9, 2), (10, 1), (11, 1)
(*ix*) (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1) (*x*) ϕ.
18. 161, 168 19. 20, 8.

20. It is not possible to distribute the money in such a way that each gets full rupees.
21. 40 years. 22. 95. 23. 4, 5, 6, 7, 8, 9 metres.
24. Krishna gets 2, 3, 4, 5, 6, 7 toffees and correspondingly Sita gets 2, 4, 6, 8, 10, 12 toffees.
25. 963.

CHAPTER 2

Page 63.

1. True : (*i*), (*iii*), (*iv*), (*viii*), (*ix*), (*x*).
   False : (*ii*), (*v*), (*vi*), (*vii*).
2. (*i*) {1, 2, 3, 4, 6, 12} (*ii*) {1, 2, 3, 4, 6, 8, 12, 16, 24, 48}
   (*iii*) {1, 2, 4, 5, 10, 20, 25, 50, 100} (*iv*) {1, 41}
   (*v*) {1, 5, 25, 125} , (*vi*) {1, 71}
   (*vii*) {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 25, 30, 50, 60, 75, 100, 150, 300}
   (*viii*) {1, 61} (*ix*) {1, 3, 41, 123}
   (*x*) {1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 48, 60, 80 120, 240}.
4. (*i*) {2, 4, 6,...} (*ii*) {5, 10, 15,...} (*iii*) {7, 14, 21,...}
   (*iv*) {6, 12, 18,...} (*v*) {3, 6, 9,...} (*vi*) {4, 8, 12,...}
   (*vii*) {11, 22, 33,...} (*viii*) {9, 18, 27,...} (*ix*) {10, 100, 1000,...}
   (*x*) N.
5. The least of every set in Exercise 2 is 1. The greatest of each set is the number itself.

   None of the sets of Exercise 4 has a greatest. The least in each case is the given number.

Page 68.

(*ii*), (*iv*), (*v*), (*vi*), (*ix*), (*x*).

Page 68.

1. (*ii*), (*v*).
2. (*i*), (*iii*), (*v*).

Page 69.

(*ii*), (*iv*), (*vii*).

Page 69.

(*i*), (*ii*), (*v*).

Page 69.

True : (*i*), (*ii*), (*v*), (*viii*), (*ix*)
False : (*iii*), (*iv*), (*vi*), (*vii*), (*x*).

Page 70.

(*ii*), (*iii*), (*vi*).

Page 70.

(*ii*), (*v*), (*vi*).

Page 71.

True : (*i*), (*iii*), (*v*), (*vii*), (*ix*), (*x*)
False : (*ii*), (*iv*), (*vi*), (*viii*).

Page 71.

1. (*i*), (*iv*), (*vi*), (*ix*).
2. True : (*iii*), (*iv*), (*vi*), (*viii*), (*ix*)
   False : (*i*), (*ii*), (*v*), (*vii*), (*x*).

Pages 72-73.

3. Only such natural number is 1.
4. No 5. {2} 6. True.
7. (*i*) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.
8. (*i*) Nil (*ii*) One (*iii*) Three (*iv*) Four (*v*) Six (*vi*) Ten.

Page 77.

(*i*) 4 (*ii*) 15 (*iii*) 28 (*iv*) 7 (*v*) 1 (*vi*) 4

Page 81.

(*i*) 141 (*ii*) 199 (*iii*) 5 (*iv*) 84.

Page 84.

(*i*) 1 (*ii*) 42.

Page 86.

(*ii*), (*v*).

Page 88.

1. (*i*) 24 (*ii*) 18 (*iii*) 8 (*iv*) 154 (*v*) 77
   (*vi*) 28 (*vii*) 60 (*viii*) 120 (*ix*) 40 (*x*) 168
2. (*i*) 20 (*ii*) 42 (*iii*) 30.

Page 91.

(*i*) 3780 (*ii*) 2520 (*iii*) 80.

Page 93.

(*i*) $3^3 \times 5^2$ (*ii*) $2^4 \times 3 \times 11$ (*iii*) $2 \times 3^2 \times 5 \times 11$
(*iv*) 2^{10} (*v*) $2^2 \times 3 \times 5 \times 11$ (*vi*) $2^4 \times 5^3 \times 13$
(*vii*) $2 \times 3^4 \times 5^2$ (*viii*) $2^2 \times 3 \times 5 \times 11 \times 17$ (*ix*) $2^4 \times 3^4 \times 7 \times 11$
(*x*) $2^6 \times 3^2 \times 7^2 \times 31$.

Page 94.

1. (*i*) 198 (*ii*) 273 (*iii*) 1 (*iv*) 123 (*v*) 1
(*vi*) 1 (*vii*) 1 (*viii*) 21 (*ix*) 2 (*x*) 61.

2. (*i*) 924 (*ii*) 18900 (*iii*) 101551200 (*iv*) 12600 (*v*) 630
(*vi*) 279734 (*vii*) 3168 (*viii*) 1680 (*ix*) 493284
(*x*) 165672.

REVIEW EXERCISES : PAGES 95-97

1. (*ii*), (*iii*), (*iv*).

2. One such set of numbers is {24, 25, 26, 27, 28}.

3. 1.

5. One of the numbers should be even and the other odd.

7. 1064, 3318 10. 60, 140 11. 100, 126 12. 27, 99 16. One, 19.1.

20. The remainder can be 1 or 4 ; or 5 could also be a factor of the number.

23. 1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47, 49.

26. I (*i*) 27, 45 (*ii*) 126, 234 (*iii*) 240, 312 (*iv*) 12, 408
(*v*) 75, 105 (*vi*) 36, 60 (*vii*) 72, 96.
II (*i*) 144, 450 (*ii*) 36, 42 (*iii*) No such numbers exist.
(*iv*) 18, 150 (*v*) 20, 42.

Note. The pairs of numbers determined are not unique.

29. 28.

30. (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73).

CHAPTER 3

Page 107.

1. (*i*) $\frac{2}{5}$ (*ii*) $\frac{2}{3}$ (*iii*) $\frac{7}{11}$
(*iv*) $\frac{3}{7}$ (*v*) $\frac{123}{1300}$ (*vi*) $\frac{20}{33}$
(*vii*) $\frac{12}{7}$ (*viii*) $\frac{77}{240}$ (*ix*) $\frac{57}{100}$.

2. None.

3. (*i*) 3 (*ii*) 48 (*iii*) 6.

4. (i) $\frac{1}{6}$ (ii) $\frac{1}{a}$ (iii) $\frac{b}{3a}$

(iv) $\frac{x}{y}$ (v) $\frac{y}{x^2}$ (vi) $\frac{b^2}{3a^2}$

(vii) $\frac{6x}{13ac}$ (viii) $\frac{2c}{3a}$ (ix) $\frac{12m}{5an}$.

5. (i) $\frac{x}{y}$ (ii) $\frac{2+m}{2+n}$ (iii) $\frac{2x}{3y}$

(iv) $\frac{1+2x}{1}$ (v) $\frac{a}{b}$ (vi) $\frac{1}{2}$

(vii) $\frac{3a+2b+5c}{4(2a+b+3c)}$ (viii) $\frac{5z}{1}$.

6. $\frac{112}{308}$, $\frac{116}{319}$, $\frac{120}{330}$, $\frac{124}{341}$.

7. $\frac{35}{63}$, $\frac{50}{90}$, $\frac{60}{108}$.

Pages 109-110.

1. (i) $\frac{22}{15}$ (ii) $\frac{22}{15}$ (iii) $\frac{127}{72}$ (iv) $\frac{127}{72}$.

2. (i) $\frac{1451}{420}$ (ii) $\frac{1451}{420}$ (iii) $\frac{23}{12}$ (iv) $\frac{23}{12}$.

Pages 112-113.

1. (i) $\frac{8}{15}$ (ii) $\frac{8}{15}$ (iii) $\frac{77}{104}$ (iv) $\frac{77}{104}$

(v) $\frac{135}{308}$ (vi) $\frac{135}{308}$ (vii) $\frac{18}{91}$ (viii) $\frac{18}{91}$.

2. (i) $\frac{5}{6}$ (ii) $\frac{5}{6}$ (iii) $\frac{35}{504}$ (iv) $\frac{635}{504}$.

3. (i) $\frac{ab}{2}$ (ii) $\frac{a}{b}$ (iii) $\frac{3x}{4y}$ (iv) $\frac{a^2b^3}{1}$

(v) $\frac{5x^3y^8}{3}$ (vi) $\frac{x^3y^4z}{2}$ (vii) $\frac{a^2b^4c^2}{1}$ (viii) $\frac{a^2b^4c^2}{1}$

(ix) $\frac{17y^2z^3}{9}$ (x) $\frac{17y^2z^2}{9}$ (xi) $\frac{7a^2b^3}{1}$.

Page 116.

1. (*i*) $\frac{11}{7}$ (*ii*) $\frac{17}{12}$ (*iii*) $\frac{27}{22}$ (*iv*) $\frac{3}{2a}$

(*v*) $\frac{6b}{7a}$ (*vi*) $\frac{3a}{2}$ (*vii*) $\frac{1}{a}$ (*viii*) $\frac{b}{1}$

(*ix*) $\frac{b^3}{a^3}$.

2. (*i*) $\frac{14a^3}{15}$ (*ii*) $\frac{56n^4}{15m}$ (*iii*) $\frac{6bx}{ay}$ (*iv*) $\frac{5b}{4}$

(*v*) $\frac{15}{14}$ (*vi*) $\frac{4x^3}{3y^3}$.

Page 117.

(*i*) $\frac{a^2+6b^2}{9b^2}$ (*ii*) $\frac{b+a}{1}$ (*iii*) $\frac{1}{1}$ (*iv*) $\frac{(ad+bc)^2}{b^3d^2}$.

Pages 119-120.

1. (*i*) $>$ (*ii*) $>$ (*iii*) $<$ (*iv*) $=$ (*v*) $<$ (*vi*) $=$
2. (*i*) $<$ (*ii*) $<$ (*iii*) $>$ (*iv*) $>$ (*v*) $>$ (*vi*) $>$
3. (*i*) $>$ (*ii*) $<$ (*iii*) $<$ (*iv*) $>$ (*v*) $<$ (*vi*) $<$
4. (*i*) $<$ (*ii*) $<$ (*iii*) $>$ (*iv*) $>$ (*v*) $<$ (*vi*) $<$
5. (*i*) $<$ (*ii*) $<, <$ (*iii*) $<$ (*iv*) $<, <$ (*v*) $<$.

   (*vi*) $<, <$.
6. (*i*) $\left\{\frac{1}{13}, \frac{1}{11}, \frac{1}{8}, \frac{1}{7}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$,

   $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{7}, \frac{1}{8}, \frac{1}{11}, \frac{1}{13}\right\}$

   (*ii*) $\left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{14}{15}\right\}$, $\left\{\frac{14}{15}, \frac{11}{12}, \frac{7}{8}, \frac{5}{6}, \frac{3}{4}, \frac{1}{2}\right\}$

   (*iii*) $\left\{\frac{4}{2}, \frac{3}{1}, \frac{6}{1}, \frac{7}{1}, \frac{8}{1}\right\}$, $\left\{\frac{8}{1}, \frac{7}{1}, \frac{6}{1}, \frac{3}{1}, \frac{4}{2}\right\}$.

Page 126.

(*i*), (*ii*), (*v*), (*vi*).

Page 127.

1. (*i*) ·65 (*ii*) ·46875 (*iii*) ·056

   (*iv*) ·076 (*v*) ·053125 (*vi*) 1·35625.
2. (*i*) $\frac{81}{250}$ (*ii*) $\frac{20123}{10000}$ (*iii*) $\frac{549}{20}$

   (*iv*) $\frac{2123}{1000}$ (*v*) $\frac{1357}{10000}$ (*vi*) $\frac{155617}{5000}$.

Page 129.

1. 1·587, 2·374, 2·476, 3 273, 3·365, 3·373.
2. (*i*) $>$ (*ii*) $<$ (*iii*) $>$ (*iv*) $>$.

Pages 139-141.

1. (*i*) $\frac{1}{2}x^2 + \frac{11}{2}x + 15$ (*ii*) $\frac{1}{2}x^2 + \frac{841}{840}xy + \frac{1}{2}y^2$

   (*iii*) $\cdot005x^2 + \cdot0015xy + \cdot185xz + \cdot0555yz$

   (*iv*) $\frac{1}{3}z^2 + \frac{5}{12}z + \frac{1}{8}$ (*v*) $\cdot65xy + \frac{1}{3}xz + 3\cdot9y^2 + 2yz$

   (*vi*) $\frac{1}{2}x^3 + \frac{7}{6}x^2 + \frac{5}{13}x + \frac{35}{39}$

   (*vii*) $x + \cdot5y + 3\cdot5z$ (*viii*) $\cdot3x^3 + \frac{3}{14}y^3 + \frac{1}{7}x^2y + \cdot45xy^2$

   (*ix*) $\frac{2}{3}x^2yz^2 + \frac{5}{4}xy^2z^2 + \frac{7}{3}xyz$

   (*x*) $\frac{2}{9}xy^2z^2 + \frac{14}{5}x^2yz^2 + 4x^2y^2z$.

3. (*i*) $\frac{xy}{x+y}$ (*ii*) $\frac{y(2xy^2+3)}{3xy^3+3}$ (*iii*) $\frac{3x+2y}{2x+5y}$

   (*iv*) $\frac{x(xy+5)}{10}$ (*v*) $\frac{x(x+1)}{x^2+1}$ (*vi*) $\frac{18y}{7}$

   (*vii*) $\frac{2x^2+3}{4x+3}$ (*viii*) $\frac{2y^2+1}{3y}$ (*ix*) $\frac{y^2+1}{2y}$

   (*x*) $\frac{y^3}{84a^2}$ (*xi*) $\frac{2(2x+3)}{x(x+2)}$ (*xii*) $\frac{2xy}{3}$

   (*xiii*) $\frac{91}{88}$ (*xiv*) x^2y^2 (*xv*) $\frac{15}{xy}$

   (*xvi*) $\frac{44}{49xy}$.

Pages 142-143.

1. (*i*) $\frac{2}{7}$ (*ii*) $\frac{14}{15}$ (*iii*) $\frac{3}{2}$ (*iv*) $\frac{133}{85}$

   (*v*) $\frac{4}{9}$ (*vi*) $\frac{67}{22}$ (*vii*) $\frac{41}{12}$ (*viii*) $\frac{1}{2}$

   (*ix*) $\frac{3}{7}$ (*x*) 5 (*xi*) $\frac{3}{17}$ (*xii*) $\frac{4}{5}$

   (*xiii*) $\frac{3}{2}$ (*xiv*) $\frac{9}{4}$ (*xv*) $\frac{2}{5}$.

2. (*i*) $\{1\}$ (*ii*) ϕ (*iii*) $\left\{\frac{7}{4}\right\}$ (*iv*) ϕ

(*v*) ϕ (*vi*) $\{2\}$ (*vii*) $\{2\}$ (*viii*) ϕ

(*ix*) ϕ (*x*) ϕ (*xi*) $\left\{\frac{19}{2}\right\}$.

3. (*i*) $\left\{\frac{6}{7}\right\}$ (*ii*) $\left\{\frac{39}{88}\right\}$ (*iii*) $\left\{\frac{1}{2}\right\}$ (*iv*) $\left\{\frac{25}{8}\right\}$

(*v*) $\{25\}$ (*vi*) $\{4\}$ (*vii*) ϕ (*viii*) ϕ.

4. (*i*) $\left\{x : \frac{2}{3} < x < 2\right\}$ (*ii*) $\left\{x : x < \frac{3}{10}\right\}$ (*iii*) $\left\{x : x > \frac{15}{2}\right\}$

(*iv*) $\left\{x : x < 6\right\}$ (*v*) $\left\{x : x > \frac{1}{12}\right\}$ (*vi*) ϕ

(*vii*) $\left\{x : x \leqslant \frac{3}{2}\right\}$ (*viii*) ϕ (*ix*) $\left\{x : x < \frac{13}{48}\right\}$

(*x*) $\left\{x : x \geqslant \frac{1}{2}\right\}$ (*xi*) $\left\{x : x \leqslant \frac{3}{2}\right\}$ (*xii*) $\left\{x : x \leqslant \frac{3}{8}\right\}$

(*xiii*) $\left\{x : x \geqslant \frac{1}{2}\right\}$ (*xiv*) ϕ (*xv*) $\left\{x : x \geqslant \frac{3}{5}\right\}$.

Pages 145-146.

(1) 12 (2) 30, 80 (3) $\frac{27}{2}$ (4) 75, 35

(5) $\frac{41}{56}$ (6) Rs. 137.50 (7) $7\frac{1}{3}$ (8) $\frac{7}{15}$

(9) 5 p.m. (10) 16 days (11) $1\frac{13}{47}$ hours (12) 24 days

(13) $16\frac{2}{3}$ litres (14) Rs. $5,555\frac{5}{9}$ and Rs. $4,444\frac{4}{9}$

(15) $X : 250$, $Y : 300$, $Z : 240$.

REVIEW EXERCISES : PAGES 146-149

1. (*i*) $\frac{11}{2}$, $\frac{3}{5}$ (*ii*) 5, $\frac{6}{10}$ (*iii*) $\frac{11}{2}$, $\frac{3}{5}$ (*iv*) 3, $\frac{3}{5}$.

2. (*i*) None exists
(*ii*) least 1, greatest does not exist
(*iii*) greatest 2, least does not exist
(*iv*) greatest 2, least 1
(*v*) least 1, greatest does not exist
(*vi*) greatest 2, least does not exist
(*vii*) greatest 2, least 1.

8. (*i*) $a^2x^2y^2 + abx^2z^2 + aby^4 + b^2y^2z^2$

(*ii*) $ax^2 + by^2 + cz^2 + (a + b)\,xy + (a + c)\,xz + (b + c)\,yz$

(*iii*) $\cdot5ax + ay + 1\cdot5az + \frac{1}{4}bx + \frac{3}{4}by + bz$

(*iv*) $1\cdot7ax + 2\cdot3ay + \cdot68xz + \cdot92yz$

(*v*) $3xy^2z + 2xyz^2 + \frac{1}{7}\,x^2y^2z^2$.

9. (*i*) $\frac{7}{3y}$ (*ii*) x (*iii*) $\frac{5}{3}$ (*iv*) $\frac{3}{a}$

(*v*) $\frac{z}{xy}$ (*vi*) $\frac{ab}{z}$ (*vii*) $\frac{1}{20}$ (*viii*) $\frac{x}{y}$

(*ix*) xz (*x*) $\frac{5ax}{3cz}$.

10. (*i*) $\left\{\frac{6}{5}\right\}$ (*ii*) $\left\{\frac{7}{5}\right\}$ (*iii*) $\{28\}$

(*iv*) $\left\{\frac{1728}{845}\right\}$ (*v*) ϕ (*vi*) $\left\{x : x < \frac{87}{40}\right\}$

(*vii*) $\{x : x < 2\}$ (*viii*) ϕ (*ix*) $\left\{x : 15 < x \leqslant \frac{35}{2}\right\}$

(*x*) F.

11. 160, 440 12. 160 13. 6 minutes

14. $2\frac{2}{11}$ hours 15. 2 km. p.h. 16. Rs. 2800, Rs 2000

17. Rs. 800, Rs. 1300 18. Rs. 9900 19. Rs. 63

20. Rs. 600, Rs. 400.

CHAPTER 4

Page 156.

(*i*) 100 (*ii*) + 100 (*iii*) − 100 (*iv*) + 100

(*v*) + 1400 (*vi*) − 1400 (*vii*) + 1400 (*viii*) − 1400

(*ix*) − 1 (*x*) + 1 (*xi*) − 1 (*xii*) + 1

(*xiii*) $+\frac{11}{12}$ (*xiv*) $-\frac{11}{85}$ (*xv*) + (*xvi*) $-\frac{13}{8}$

(*xvii*) − 5 (*xviii*) − 5 (*xix*) $+\frac{25}{12}$ (*xx*) $+\frac{25}{12}$.

Pages 157-158.

1. (*i*) $+5$ (*ii*) -23 (*iii*) $+\frac{27}{28}$ (*iv*) $-\frac{59}{40}$

(*v*) $-11{\cdot}8$ (*vi*) $+2{\cdot}06$ (*vii*) $+2{\cdot}9$ (*viii*) 0

(*ix*) $-1{\cdot}4$ (*x*) -5 (*xi*) 0 (*xii*) $+\frac{7}{3}$

(*xiii*) $-\frac{8}{9}$ (*xiv*) $+\frac{1}{15}$.

2. (*i*) -6 (*ii*) $-\frac{3}{2}$ (*iii*) $-8{\cdot}85$ (*iv*) $+\frac{1}{6}$.

3. (*i*) -40 (*ii*) $+1{\cdot}8$ (*iii*) $+\frac{11}{12}$.

Page 161.

1. (*i*) -3 (*ii*) $+\frac{7}{3}$ (*iii*) $+2{\cdot}25$ (*iv*) $+2$

(*v*) $+\frac{17}{12}$ (*vi*) $-\frac{7}{3}$.

3. The result is zero in both the cases.

Pages 163-164.

1. (*i*) $+10$ (*ii*) -21 (*iii*) $+\frac{2}{3}$ (*iv*) $+{\cdot}2$ (*v*) $-\frac{5}{3}$.

2. (*i*) $+9$, $+5$, $+15$ (*ii*) $-\frac{217}{120}$, $-\frac{343}{120}$, $-\frac{313}{120}$

(*iii*) $-4{\cdot}65$, $+2{\cdot}15$, $+{\cdot}05$.

3. (*i*) 8 (*ii*) $\frac{29}{15}$ (*iii*) $\frac{8}{3}$ (*iv*) $\frac{11}{35}$.

Pages 167-168.

1. (*i*) -275 (*ii*) $-\frac{21}{32}$ (*iii*) $-{\cdot}51$ (*iv*) $-6{\cdot}08$

(*v*) -1 (*vi*) $-\frac{3}{14}$.

2. (*i*) -120 (*ii*) $-\frac{2}{5}$ (*iii*) 0 (*iv*) $+4{\cdot}2$

(*v*) $+18$.

3. (*i*) -21 (*ii*) $+18$ (*iii*) $+\frac{8}{5}$ (*iv*) $-\frac{2}{7}$.

4. (i) $-\frac{1}{6}$ (ii) -40 (iii) $+3$.

5. (i) $+6$ (ii) -2 (iii) -3 (iv) $-\frac{1}{2}$

(v) $-\frac{1}{4}$ (vi) $+\frac{7}{3}$.

Page 171.

1. (i) $-\frac{1}{3}$ (ii) $-\frac{3}{2}$ (iii) $+\frac{8}{7}$ (iv) $+\frac{25}{58}$

(v) $-\frac{4}{13}$ (vi) $-\frac{20}{7}$ (vii) $-\frac{4}{5}$ (viii) $+\frac{5}{4}$

(ix) $-\frac{20}{141}$ (x) $-\frac{1}{8}$ (xi) $+1$ (xii) -1

(xiii) $-\frac{15}{8}$ (xiv) $-\frac{24}{7}$ (xv) $+\frac{20}{3}$.

Pages 175-176.

1. (i) $-17, -9, -7, -\frac{11}{13}, 0, +0\cdot25, +3, +8$

(ii) $-3, -\frac{7}{12}, +\frac{1}{4}, +\frac{5}{6}, +\frac{9}{3}$

(iii) $-3, -0\cdot25, 0, +\frac{1}{4}, +\frac{3}{4}, +10$

(iv) $-\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}, +\frac{1}{2}, +\frac{3}{4}, +\frac{5}{6}$.

(v) $-3, -\frac{2}{3}, +\frac{8}{12}, +2\frac{1}{3}$

(vi) $-21, -12, -7, -6, 0, +2, +12, +21$

2. (ii), (iii)

Page 180.

(i), (ii), (iii), (vi), (vii).

Pages 187-188.

3. (i) $-21a^4b^2c$ (ii) $\frac{7}{4}x^4y^2z^3$ (iii) $\frac{xy}{x+y}$

(iv) $\frac{xy}{y-x}$ (v) $3xy$ (vi) $\frac{1}{3(b+c)}$

(vii) $\frac{15x^2}{2az^2}$ (viii) $y+2x$ (ix) $\frac{8a-b}{2}$

(x) $\frac{2(x+3)}{3}$ (xi) $\frac{x-y}{x+y}$ (xii) $\frac{a+b}{b(a-b)}$

(xiii) $\frac{a-b}{a+b}$ (xiv) $\frac{y^2+4y-4}{y^2-4}$ (xv) $\frac{6x}{9x^2-16}$

(xvi) $\frac{10}{49y^2-25}$ (xvii) $\frac{2a^3-3b}{6a^2b}$ (xviii) $\frac{-2x(2x+1)}{1-x^2}$

(xix) $\frac{y(2x+5y)}{(x+2y)(x+3y)}$ (xx) $\frac{2x^2+10x+3}{(x+2)(x+3)}$.

Pages 190-192.

1. (i) -14 (ii) $\frac{75}{2}$ (iii) $\frac{315}{47}$ (iv) $\frac{-115}{896}$

(v) $\frac{264}{53}$ (vi) $\frac{-7}{120}$ (vii) $\frac{341}{158}$ (viii) $\frac{84}{253}$

(ix) 2 (x) $\frac{-43}{2}$ (xi) $\frac{-1}{6}$ (xii) -3

(xiii) $\frac{-3}{2}$ (xiv) 3 (xv) Any $x \in Q$ is a solution,

2. (i) $\{x : x > -12\}$ (ii) $\left\{x : x < -\frac{11}{6}\right\}$

(iii) $\left\{x : x < -\frac{840}{143}\right\}$ (iv) $\{x : x > 2\}$

(v) $\left\{x : x \leqslant \frac{200}{3}\right\}$ (vi) $\left\{x : x \leqslant -\frac{73}{120}\right\}$

(vii) $\left\{x : x \geqslant -\frac{5}{4}\right\}$ (viii) $\{x : x \geqslant 37\}$

(ix) $\left\{x : x \geqslant -\frac{1}{2}\right\}$ (x) $\{x : x \geqslant -6\}$.

3. (i) $-\frac{3}{5}$ (ii) 10 (iii) $-\frac{31}{6}$ (iv) -5 (v) -1.

Page 194.

1. (i) $\{2, -2\}$ (ii) $\left\{\frac{5}{7}, -\frac{5}{7}\right\}$ (iii) $\{8, -2\}$ (iv) $\{11, 3\}$

(v) $\left\{\frac{5}{7}, \frac{3}{7}\right\}$ (vi) $\left(-\frac{51}{8}, -\frac{69}{8}\right)$ (vii) $\{-4, -10\}$ (viii) 8, $\left\{\frac{6}{5}\right\}$

(ix) $\{5\}$ (x) ϕ.

2. (i) $\{4, -4\}$ (ii) $\{0\}$ (iii) $\{4, 0, -4\}$ (iv) ϕ.

2. (*i*) $\{4, -4\}$ (*ii*) $\{0\}$ (*iii*) $\{4, 0, -4\}$ (*iv*) ϕ.

3. (*i*) $\{x : -5 < x < 5, x \in \mathbf{Q}\}$ (*ii*) $\{x : -3 < x < \frac{1}{3}, x \in \mathbf{Q}\}$

(*iii*) $\left\{x : \frac{9}{7} < x < \frac{25}{7}, x \in \mathbf{Q}\right\}$

(*iv*) $\{x : x > 2, x \in \mathbf{Q}\} \cup \{x : x < -2, x \in \mathbf{Q}\}$

(*v*) $\{x : 4 < x < 10, x\ \mathbf{Q}\}$

(*vi*) $\{x : x > 13, x \in \mathbf{Q}\} \cup \{x : x < 5, x \in \mathbf{Q}\}$

(*vii*) $\left\{x : x > \frac{16}{3}, x \in \mathbf{Q}\right\} \cup \left\{x : x < -\frac{8}{3}, x \in \mathbf{Q}\right\}$

(*viii*) $\left\{x : x > -\frac{32}{35}, x \in \mathbf{Q}\right\} \cup \left\{x : x < -\frac{52}{35}, x \in \mathbf{Q}\right\}$

(*ix*) ϕ

(*x*) The set $\mathbf{Q}$ except the number $\frac{5}{2}$.

REVIEW EXERCISES : PAGES 194-197

1. (*i*) 2·4, − 3·75 (*ii*) 2·16, − 3·75 (*iii*) 2·4, − 3·75
(*iv*) − ·7, − 3·75.

2. ***A*** : least − 5, greatest does not exist.
B : greatest − 3, least does not exist.
C : greatest − 3, least − 5.
D : none exists.
E : greatest 0, least does not exist
F : none exists.
G : greatest 0, least − 1.
H : least − 1, greatest does not exist.

9. (*i*) $\frac{1}{2x}$ (*ii*) 8 (*iii*) $\frac{9a^2 - 4b^2}{a^2 - b^2}$ (*iv*) $-\frac{8}{7}$ (*v*) $-x^2$.

10. (*i*) $\frac{1}{4}$ (*ii*) $\frac{297}{59}$ (*iii*) $\frac{390}{223}$ (*iv*) no solution

(*v*) $-\frac{55}{61}$ (*vi*) $\frac{321}{130}$ (*vii*) $-\frac{337}{314}$ (*viii*) $\frac{7}{5}$

(*ix*) $-\frac{34}{7}$ (*x*) no solution (*xi*) $\frac{ad - bc}{(a + d) - (b + c)}$

(*xii*) $\frac{bc - ad}{(a + d) - (b + c)}$.

11. (i) $\{x : x < 0\}$ (ii) $\left\{x : x > \frac{91}{51}\right\}$ (iii) $\left\{x : x < -\frac{35}{22}\right\}$

(iv) $\left\{x : x \geqslant -\frac{81}{112}\right\}$ (v) $\left\{\frac{13}{2}, -\frac{5}{2}\right\}$ (vi) $\left\{1, -\frac{11}{3}\right\}$

(vii) $\{5, -5, 1, -1\}$ (viii) $\{6, -6, 0\}$ (ix) ϕ

(x) $\{1, -1\}$ (xi) $\left\{x : -\frac{2}{3} < x < \frac{14}{15}\right\}$

(xii) $\left\{x : x > \frac{23}{60}\right\} \cup \left\{x : x < -\frac{7}{60}\right\}$

(xiii) $\left\{x : -\frac{19}{5} < x < \frac{16}{5}\right\}$

(xiv) $\left\{x : x > \frac{1}{56}\right\} \cup \left\{x : x < -\frac{13}{56}\right\}$

(xv) $\{x : x > 3\} \cup \{x : x < 2\}$

(xvi) $\{x : 2 < x < 3\}$ (xvii) $\{2, 3\}$

(xviii) $\{x : x < 1\} \cup \{x : x > 20\}$ (xix) $\{x : 0 < x < 1\}$

(xx) $\{0, 1\}$.

CHAPTER 5

Pages 200-202.

1. (i) $3x + (-2) = 0$ (ii) $\frac{3}{5}x + (-4) = 0$

(iii) $ax + (-b) = 0$ (iv) $5x + 2 = 0$

(v) $\frac{5}{4}x + \frac{1}{2} = 0$ (vi) $\cdot 5x + \left(-\frac{5}{12}\right) = 0$

(vii) $ax + (b - c) = 0$ (viii) $(a - c)x + b = 0$

(ix) $(a - c)x + (b - d) = 0$.

4. (i), (ii), (iv), (v), (vi), (vii) are linear and (iii), (viii) are not linear. The domain is the set **Q** except the numbers

(i) 0, 1 (ii) a, b (iii) $1, -3$ (iv) $-\frac{1}{2}$

(v) 7, 4 (vi) $-\frac{5}{2}$ (vii) $-5, 4$ (viii) $1, -1$.

5. (iii), (viii).

Page 202.

1. (i) $\frac{2}{3}$ (ii) 6 (iii) $\frac{b}{a}$ (iv) $-\frac{2}{5}$

(v) $-\frac{2}{5}$ (vi) $\frac{5}{6}$ (vii) $\frac{(c-b)}{a}$ (viii) $\frac{b}{(c-a)}$

(ix) $\frac{(d-b)}{(a-c)}$

2. (i) 22 (ii) $\frac{5}{3}$ (iii) 3 (iv) $\frac{1}{67}$ (v) 0 (vi) 11·8

3. (*i*) $\frac{60}{23}$ (*ii*) $-\frac{13}{5}$ (*iii*) 7 (*iv*) $\frac{7}{18}$

4. (*i*) $\frac{4}{7}$ (*ii*) $\frac{(a+b)}{2}$ (*iv*) $\frac{1}{5}$ (*v*) $-\frac{5}{4}$

(*vi*) $\frac{1}{4}$ (*vii*) $-\frac{1}{4}$.

Pages 203-204.

1. (*i*) $a = b$ (*ii*) $a = b$ (*iii*) $l + m = 0$
(*iv*) $l + m = 0$ (*v*) $ab = c$ (*vi*) $ab + c = 0$
(*vii*) $ab + c = 0$ (*viii*) $ab = c$ (*ix*) $lq + mp = 0$
(*x*) $lq + mp = 0$ (*xi*) $lq = mp$ (*xii*) $ae + bd = cd$.

2. (*i*), (*ii*) and (*iv*) consistent, (*iii*) not consistent.

3. (*i*), (*v*) consistent, (*ii*), (*iii*), (*iv*), (*vi*) not consistent.

4. (*ii*), (*iii*) consistent, (*i*), (*iv*) not consistent.

Page 205.

1. (*i*) $2x + 3y + (-4) = 0$ (*ii*) $2x + (-3y) + 4 = 0$
(*iii*) $2x + (-3)y + (-4) = 0$ (*iv*) $2x + (-3)y + (-3) = 0$
(*v*) $x + 4y + (-2) = 0$ (*vi*) $4x + 2y + 0 = 0$
(*vii*) $\frac{3}{2}x + \left(-\frac{7}{4}\right)y + \left(-\frac{7}{4}\right) = 0$
(*viii*) $\frac{19}{3}x + \frac{55}{12}y + \frac{11}{6} = 0$.

2. (*i*), (*ii*), (*v*), (*vi*) linear, (*iii*) not linear.
(*i*) y cannot be 0 (*ii*) x cannot be 5 (*iii*) y cannot be 0
(*iv*) x cannot be $-\frac{5}{2}$, y cannot be $\frac{6}{13}$
(*v*) y cannot be $\frac{1}{6}, -\frac{3}{2}$ (*vi*) y cannot be $\frac{9}{28}, -\frac{11}{7}$.

Page 207.

2. Same : (*i*), (*iv*). Different : All others.

Page 208.

1. (*i*) $\{(x, 0) : x \in Q\}$ (*ii*) $\left\{\left(x, -\frac{5}{2}\right) : x \in Q\right\}$
(*iii*) $\left\{\left(x, \frac{11}{7}\right) : x \in Q\right\}$ (*iv*) $\{(0, y) : y \in Q\}$
(*v*) $\left\{\left(-\frac{5}{3}, y\right) : y \in Q\right\}$ (*vi*) $\left\{\left(\frac{5}{8}, y\right) : y \in Q\right\}$.

2. (i) $\{(x, y) : x, y \in \mathbf{Q}, y \neq 0\}$

(ii) $\left\{\left(x, y\right) : x, y \in \mathbf{Q}, y \neq \frac{5}{2}\right\}$

(iii) $\left\{\left(x, y\right) : x, y \in \mathbf{Q}, y \neq \frac{11}{7}\right\}$

(iv) $\{(x, y) : x, y \in \mathbf{Q}, x \neq 0\}$

(v) $\left\{\left(x, y\right) : x, y \in \mathbf{Q}, x \neq -\frac{5}{3}\right\}$

(vi) $\left\{\left(x, y\right) : x, y \in \mathbf{Q}, x \neq \frac{5}{8}\right\}$.

Pages 214-215.

1. (i) $\left(\frac{29}{2}, \frac{21}{2}\right)$ (ii) $\left(\frac{5}{3}, \frac{29}{3}\right)$ (iii) $(-2, 13)$

(iv) $(4, 1)$ (v) $(6, 2)$ (vi) $(3, 2)$

(vii) $\left(\frac{3}{4}, \frac{41}{14}\right)$ (viii) $\left(-\frac{11}{3}, \frac{1}{2}\right)$ (ix) $\left(1, \frac{4}{11}\right)$.

2. (i) $\left\{\left(-\frac{4}{3}, \frac{2}{3}\right)\right\}$ (ii) $\left\{\left(\frac{4}{3}, \frac{3}{2}\right)\right\}$ (iii) $\left\{\left(\frac{31}{20}, \frac{1}{20}\right)\right\}$

(iv) $\left\{\left(\frac{18}{13}, \frac{53}{13}\right)\right\}$ (v) ϕ

(vi) $\left\{\left(h, k\right) : k = \frac{2h + k}{3}, h, k \in \mathbf{Q}\right\}$

(vii) $\left\{\left(h, k\right) : k = \frac{2h + 4}{7}, h, k \in \mathbf{Q}\right\}$ (viii) $\{(-1, 0)\}$

(ix) $\{(0, 1)\}$ (x) ϕ (xi) $\{(h, k) : k = 2h + 3, h, k \in \mathbf{Q}\}$.

3. (i) $\left\{\left(\frac{1}{8}, \frac{1}{4}\right)\right\}$ (ii) $\{(1, 1)\}$ (iii) $\left\{\left(\frac{5}{12}, \frac{50}{119}\right)\right\}$

(iv) $\left\{\left(\frac{1}{14}, \frac{1}{6}\right)\right\}$ (v) $\left\{\left(-\frac{5}{4}, -\frac{3}{2}\right)\right\}$ (vi) $\left\{\left[\frac{2}{3}, \frac{3}{2}\right]\right\}$

(vii) $\left\{\left(\frac{1}{3}, -1\right)\right\}$ (viii) $\left\{\left(2, \frac{1}{2}\right)\right\}$ (ix) ϕ

(x) $\left\{\left(h, k\right) : k = \frac{7h}{2(5h - 7)}, h, k \in \mathbf{Q}\right\}$.

4. (i) $\{(60, 40)\}$ (ii) $\{(20, 18)\}$ (iii) $\{(4, 2)\}$

(iv) $\{(18, 54)\}$ (v) $\left\{\left(\frac{687}{62}, \frac{339}{62}\right)\right\}$ (vi) $\{(3, -2)\}$

(vii) $\{(5, -5)\}$ (viii) $\{(3, -2)\}$.

Pages 218-219.

1. (*i*) $\left(-\frac{c+d}{2a}, \frac{d-c}{2b}\right)$ (*ii*) $\left(0, -\frac{c}{b}\right)$

(*iii*) $\left(-\frac{ac+bd}{a^2+b^2}, \frac{ad-bc}{a^2+b^2}\right)$ (*iv*) $\left(\frac{bd-ac}{a^2+b^2}, -\frac{bc+ad}{a^2+b^2}\right)$

(*v*) $\left\{\frac{e-c}{a-d}, \frac{cd-ae}{b(a-d)}\right\}$ (*vi*) $\left(\frac{be-cd}{a(d-b)}, \frac{e-c}{b-d}\right)$.

2. (*i*) $(1, 3)$ (*ii*) $\left(-\frac{1}{2}, 2\right)$ (*iii*) $\left(-\frac{1}{5}, \frac{19}{5}\right)$

(*iv*) $\left(\frac{1}{13}, \frac{43}{13}\right)$ (*v*) $\left(\frac{29}{13}, \frac{11}{13}\right)$ (*vi*) $\left(\frac{16}{7}, -\frac{26}{7}\right)$

(*vii*) $\left(\frac{13}{19}, \frac{62}{19}\right)$ (*viii*) $\left(-\frac{9}{17}, \frac{2}{17}\right)$.

Page 220.

Consistent : (*i*), (*iii*), (*iv*), Inconsistent : (*ii*).

Pages 221-222.

1. (*i*), (*ii*), (*iii*), (*iv*), (*v*), (*vii*), (*viii*) are linear but (*vi*) is not.

Restrictions

(*iii*) $z \neq \frac{5}{2}$ (*iv*) $y \neq \frac{7}{12}$ (*v*) $x \neq \frac{7}{5}$

(*vi*) $y \neq 2, z \neq 3$ (*vii*) $y \neq \frac{8}{3}$ (*viii*) $y \neq 2$.

2. Yes : (*i*), No : All others.

3. (*i*) $\{(x, y, 0) : x, y \in \mathbf{Q}\}$ (*ii*) $\{(0, y, z(: y, z \in \mathbf{Q}\}$

(*iii*) $\{(x, 0, z) : x, z \in \mathbf{Q}\}$ (*iv*) $\left\{\left(\frac{3}{5}, y, z\right) : y, z \in \mathbf{Q}\right\}$

(*v*) $\{(5, y, z) : y, z \in \mathbf{Q}\} \cup \left\{\left(x, -\frac{3}{2}, z\right) x: , z \in \mathbf{Q}\right\}$

(*vi*) $\{(x, y, -5) : x, y \in \mathbf{Q}\} \cup \{(x, 4, z) : x, z \in \mathbf{Q}\}$.

Pages 226-227.

1. (*i*) $\{(-30, -39, -12)\}$ (*ii*) $\{(20, 15, 22)\}$

(*iii*) $\{(-1, 3, -4)\}$ (*iv*) $\{(6, 8, 10)\}$

(*v*) $\{(a, b, c) : a = -(11+c), b = -(21+c)/3, a, b, c \in \mathbf{Q}\}$

(*vi*) ϕ

(*vii*) $\left[\left(a, b, c\right) : a = \frac{21c-146}{13}, b = \frac{8-22c}{13}, ab, c \in \mathbf{Q}\right]$

(*viii*) ϕ.

2. (*i*) $\left\{\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{4}\right)\right\}$ (*ii*) $\left\{\left(\frac{1}{2}, 1, \frac{1}{4}\right)\right\}$

(*iii*) $\left\{\left(-\frac{1}{2}, -\frac{2}{3}, -1\right)\right\}$ (*iv*) $\left\{\left(\frac{2}{3}, \frac{1}{2}, \frac{2}{5}\right)\right\}$.

Pages 232-233.

(1) 35	(2) 36	(3) 692	(4) Father 40, Son 10
(5) 36	(6) $13\frac{1}{3}$ days	(7) 5 : 1	(8) Rs. 6, Rs. 5
(9) Rs. 960, Rs. 480	(10) Rs. 15, Rs. 1·50		(11) 4 km.p.h.
(12) 25 km.p.h.	(13) 5 min., 6 min.		(14) 11, 10
(15) 9 km., 4 km.p.h.	(16) Rs. 2500		(17) Rs. 800, Rs. 750
(18) Rs. 3500, Rs. 3250	(19) Rs. 21735		(20) Rs. 11,000, Rs. 10,000.

REVIEW EXERCISES : PAGES 234-236

1. (*i*) (1, 0) (*ii*) (2, 3) (*iii*) $\left(\frac{73}{57}, \frac{62}{57}\right)$

(*iv*) no solution (*v*) $\left(-\frac{86}{163}, \frac{157}{163}\right)$ (*vi*) (6, 6).

2. Yes : (*i*), (*iv*), No : (*ii*), (*iii*).

3. (*i*) (1, − 1) (*ii*) $\left(\frac{895}{150}, \frac{895}{198}\right)$.

4. $a^2(b-c)+b^2(c-a)+c^2(a-b)=0$.

5. (*i*) $\left\{(a, b, c) : a=\frac{c-6}{11}, b=\frac{17c+19}{11}, a, b, c \in Q\right\}$.

(*ii*) ϕ (*iii*) $\left(\frac{3}{2}, -\frac{11}{14}, \frac{73}{98}\right)$ (*iv*) $\left(-\frac{5}{4}, -\frac{10}{13}, \frac{7}{52}\right)$.

6. (*i*) $\left(\frac{1}{3}, 1, \frac{1}{5}\right)$ (*ii*) $\left(1, 1, \frac{4}{3}\right)$.

7. 37
8. Asha Rs. 700, Usha Rs. 1700
9. $\frac{5}{8}$
10. 3 days, $4\frac{1}{2}$ days
11. It empties in 12 hours.
12. Rs. 17500, Rs. 12500
13. 144, 96
14. Father 33, Son 10
15. 36, 27 years
16. $1\frac{1}{2}$, $6\frac{1}{3}$ km.p.h.
17. 60, 80 km.p.h.
18. 60 c.c., 20 c.c.
19. Rs. $10166\frac{2}{3}$, Rs. 6000
20. Rs. 4320, Rs. 4050.

CHAPTER 6

Page 238.

monomials : (*iii*), (*iv*), (*vii*)

binomials : (*i*), (*v*), (*viii*)

trinomials : (*ii*), (*vi*), (*ix*).

Pages 239-240.

1. (*i*) 2, 1 ; one (*ii*) − 2, 5 ; one (*iii*) − 3, 0 ; one
(*iv*) 5, ·7 ; one (*v*) − 1·5, 2 ; one (*vi*) 7, 0 ; one
(*vii*) − 2, 0, 7 ; two (*viii*) ·5, − 7, 8 ; two (*ix*) ·7, 0, 1 ; two
(*x*) 3, − 2, 1 ; two (*xi*) 5 / 2, 1, − 3 : two (*xii*) − 8, 0, 0 ; two
(*xiii*) 1, 1, − 1 ; two (*xiv*) − 2, 5 / 3, 0 ; two (*xv*) 3, 0, − 7 ; two
(*xvi*) 1, 7, − 3, 5 ; three (*xvii*) 2, 0, 0, − 2, 5 ; four
(*xvii*) 1, 0, 0, 0, 0, 0, 0, − 1 ; seven.
2. (*xiii*), (*xvi*), (*xviii*).

Pages 240-241.

1. (*i*) $x^2 + 3x + 2$ (*ii*) $x^2 + x - 6$ (*iii*) $x^2 - 11x + 28$
(*iv*) $2x^2 + 5x + 2$ (*v*) $6x^2 + 13x + 6$ (*vi*) $30x^2 + 13x - 77$
(*vii*) $3x^2 + 11x - 4$ (*viii*) $-2x^2 + 15x - 7$ (*ix*) $3x^2 - 2x - 8$
(*x*) $-9t^2 + 3t + 2$ (*xi*) $-16t^2 + 25$ (*xii*) $-2t^2 - t + 10$
2. (*i*) $x^2 + (a + b)x + ab$ (*ii*) $x^2 + (3q - 2p)x - 6pq$
(*iii*) $y^2 + (l - 5m)y - 5lm$.

Page 242.

1. (*i*) 4 (*ii*) $\frac{9}{2}$ (*iii*) a^2 (*iv*) $\frac{1}{4}$ (*v*) $\frac{25}{4}$
(*vi*) $\frac{b^2}{4a^2}$ (*vii*) $\frac{1}{16}$ (*viii*) $\frac{49}{64}$ (*ix*) $\frac{81}{484}$

and the corresponding linear polynomial in each case is

(*i*) $x - 2$ (*ii*) $x + \frac{3}{2}$ (*iii*) $x - a$ (*iv*) $x - \frac{1}{2}$
(*v*) $x - \frac{5}{2}$ (*vi*) $x + \frac{b}{2a}$ (*vii*) $x - \frac{1}{4}$ (*viii*) $x + \frac{7}{8}$
(*ix*) $x - \frac{9}{22}$.

2. (*i*) 2 (*ii*) $\frac{19}{4}$ (*iii*) $\frac{25}{3}$ (*iv*) $\frac{17}{4}$

(*v*) $\frac{l^2 - 4m}{4}$ (*vi*) $m - l^2$.

3. (*i*) $6x$; $x + 3$ or $-6x$; $x - 3$

(*ii*) $3x$; $x + \frac{3}{2}$ or $-3x$; $x - \frac{3}{2}$.

4. (*i*) 2 or -6 (*ii*) $\frac{19}{5}$ or $\frac{31}{5}$

(*iii*) $2m - l$ or $-2m - l$ (*iv*) $l + m$ or $m - l$.

Page 247.

(*i*), (*ii*), (*iv*), (*vi*), (*vii*), (*viii*), (*x*), (*xii*), (*xiii*), (*xv*).

(*i*) $(x + 2)(x + 3)$ (*ii*) $(x - 1)(x - 8)$ (*iv*) $(x + 4)(x - 2)$
(*vi*) $(x - 5)(x + 2)$ (*vii*) $(2x - 5)(x - 1)$ (*viii*) $(3x + 2)(x + 2)$
(*x*) $(2x - 5)(5x + 1)$ (*xii*) $(x + 2)(8x - 3)$ (*xiii*) $(3x + 4)^2$
(*xv*) $(x + 2)(7x + 2)$.

Pages 249-250.

(*i*) $(x + 1)(x + 5)$ (*ii*) $(x - 1)(x - 2)$ (*iii*) $(x + 5)(x - 1)$
(*iv*) $(x - 14)(x + 2)$ (*v*) $(2x + 5)(x + 1)$ (*vi*) $(3x - 2)(2x - 3)$
(*vii*) $(y - 1)(3y - 2)$ (*viii*) $(3y - 5)(4y + 3)$ (*ix*) $(y - 5)(2y + 5)$
(*x*) $(2t - 3)(6t - 7)$ (*xi*) $(3t - 1)(t - 2)$ (*xii*) $(t - 3)(3t + 2)$
(*xiii*) $(2u + 1)(7u - 11)$ (*xiv*) $(3 - u)(7 + u)$ (*xv*) $(3u - 5)^2$.

Page 256.

1. (*i*) $\{3, -3\}$ (*ii*) $\{2\}$ (*iii*) $\left\{-\frac{7}{2}\right\}$ (*iv*) $\{7, -12\}$

(*v*) $\{-2, -8\}$ (*vi*) ϕ (*vii*) ϕ (*viii*) $\{3, 5\}$

(*ix*) $\left\{1, -\frac{2}{3}\right\}$ (*x*) $\left\{\frac{4}{3}, -5\right\}$ (*xi*) ϕ (*xii*) ϕ.

2. (*i*) no solution (*ii*) 9 (*iii*) $-\frac{11}{15}$ (*iv*) no solution.

Page 258.

(*i*) $\{x : 1 < x < 2\}$ (*ii*) $\{x : -1 \leqslant x \leqslant 3\}$

(*iii*) $\{x : -2 < x < 3\}$

(*iv*) All rational numbers except those between -5, 4.

(*v*) All rational numbers except the numbers $1/2$, 2 and those between $1/2$ and 2.

(*vi*) $\left\{x : -\frac{5}{4} < x < -\frac{2}{3}\right\}$.

(*vii*) All rational numbers except those between 4 and 5.

(*viii*) $\{x : -5 < x < 4\}$ (*ix*) $\{x : -\frac{1}{2} < x < \frac{2}{3}\}$

(*x*) $\{x : 1 < x < \frac{9}{5}\}$ (*xi*) $\{x : -2 < x < 4\}$.

(*xii*) All rational numbers except the numbers $-3, 5$ and those between -3 and 5.

Page 259.

(1) 1 or 5 (2) 5, 11 (3) 2 or $-\frac{3}{2}$

(4) base : 4 or 6 cm. height : 6 or 4 cm.

(5) 15 cm., 7 cm. (6) 12 metres, $7\frac{1}{2}$ metres.

REVIEW EXERCISES : PAGES 260-261

1. (*ii*), (*iv*), (*vi*).
   (*ii*) $(x + 3)(x + 6)$ (*iv*) $(2x + 5)(5x - 3)$ (*vi*) $(7x + 4)(x + 2)$

2. (*i*) $\{-2, -10\}$ (*ii*) $\left\{-2, \frac{5}{3}\right\}$ (*iii*) $\{-2, 3\}$
   (*iv*) $\left\{-\frac{5}{6}, \frac{5}{2}\right\}$ (*v*) $\left\{0, \frac{1}{7}\right\}$ (*vi*) ϕ
   (*vii*) $\left\{\frac{1}{2}\right\}$ (*viii*) ϕ (*ix*) $\left\{\frac{5}{2}, -\frac{5}{2}\right\}$.

3. (*i*) 1, -4 (*ii*) no solution (*iii*) no solution (*iv*) 3.

4. (*i*) $(x + a - 1)(x - a + 2)$ (*ii*) $(2x + 3a)(2x + 3a - 4)$
   (*iii*) $(2x + 3a)(2x + 3a - 4)$ (*iv*) $(2x + a - 1)(x - 4a + 4)$.

(5) length : 16 cm., width : 4 cm., side of the square : 8 cm.

(6) base : 15 or 7 cm., height : 7 or 15 cm.

(7) 17, 18, 19 (8) 12, 8 (9) 18 hours (10) one hour.

APPENDIX

Pages 265-266.

1. (*i*) 123 (*ii*) 592 (*iii*) 540 (*iv*) 2558
   (*v*) 7272 (*vi*) 2400 (*vii*) 242756 (*viii*) 146522
   (*ix*) 955565.

2. (*i*) $(101101)_2$ (*ii*) $(3025)_6$ (*iii*) $(10111)_3$ (*iv*) $(3112)_4$
   (*v*) $(33440)_5$ (*vi*) $(441316)_7$ (*vii*) $(10534)_8$ (*viii*) $(1005345)_9$
   (*ix*) $(69150)_{11}$ (*x*) $(152319)_{12}$.

3. (*i*) $(10001110)_2$ (*ii*) $(440)_5$ (*iii*) $(33)_6$ (*iv*) $(60)_7$
   (*v*) $(1157)_9$ (*vi*) $(2991)_{12}$ (*vii*) $(125713)_{11}$ (*viii*) $(111001)_2$
   (*ix*) $(103430)_7$ (*x*) $(311)_8$.

Page 267.

1. (*i*) 11010 (*ii*) 111111 (*iii*) 1111110 (*iv*) 1100100
   (*v*) 11010 (*vi*) 1001.

2. (*i*) 111, 1000, 1010, 1100 (*ii*) 11, 100, 101, 111
(*iii*) 1000, 1001, 1010, 1100.

3, (*i*) $>$ (*ii*) $<$ (*iii*) $<$ (*iv*) $>$.

TEST PAPER I

Pages 268-269.

1. (*a*) $\{2, 0, 3, 7, 4, 8, 9, 6, 11\}$, $\{0, 7, 8\}$.

(*b*) For example, $-\frac{2}{3}$, ·05, 13·24, $-$ 2·48, $\frac{13}{5}$ are rational numbers but not integers and 0, $-$ 7, $-$ 24, $-$ 13, $-$ 41 are integers but not natural numbers.

2. $\{1, 2, 3, 4, 6, 8, 12, 24\}$, $\{1, 2, 3, 6, 7, 14, 21, 42\}$, $\{1, 2, 3, 6\}$, 1, 6 6.

4. (*b*) $2^{10} \times 3^{3}, 2^{7} \times 5^{2}$. $2^{10} \times 3^{3} \times 5^{2}$.

5. (*a*) ϕ. However, if the domain is taken as **Q**, the truth set becomes $\left\{-\frac{1}{3}\right\}$. (*b*) $\{1, 3, 5\}$.

7, (*b*) (*i*) $\left\{-\cdot75, 0, \frac{2}{3}, \frac{13}{15}, 1\cdot25\right\}$

(*ii*) $\{-3\cdot24, -2\cdot5, -1\cdot27, \cdot04, \cdot75\}$

8. (*a*) $\left\{\frac{7}{2}, -\frac{1}{2}\right\}$ (*b*) $\left\{\left(\frac{2}{31}, \frac{71}{31}\right)\right\}$.

9. (*a*) $am + bl = 0$ (*b*) $(x + 2)(8x - 3)$

10. (*a*) $\left\{\frac{1}{2}, \frac{20}{3}\right\}$ (*b*) 100, 80.

TEST PAPER II

Pages 269-271.

1. (*a*) ϕ, $\{4\}$, $\{4, 5\}$.

(*b*) For example, $\frac{3}{5}$, $\frac{11}{4}$, 7·34, 2·75, ·36 are fractions but not integers and 0, $-$ 3, $-$ 24, $-$ 5, $-$ 10 are integers but not fractions.

2. $\{4, 8, 12, 16, \ldots\}$, $\{6, 12, 18, 24, \ldots\}$. Infinite.
$\{12, 24, 36, 48, \ldots\}$. No. 12. 12.

5. (*a*) $\{3\}$. No change if the domain is taken as N or F. But the truth set becomes ϕ if the domain becomes the set of even natural numbers.

(*b*) $\{(1, 3), (4, 1)\}$. Finite. No.

7. (*b*) (*i*) $\left\{\frac{3}{4}, \frac{1}{2}, 0, -\frac{1}{4}, -\frac{5}{6}\right\}$

(*ii*) {·77, ·74, 0, − ·73, − ·79}.

8. (*b*) Not expressible.

9. (*a*) $a^2(b-c)+b^2(c-a)+c^2(a-b)=0$ (*b*) ϕ.

10. (*a*) $\frac{6}{7}$ (*b*) 4, 7, 9, 40 years

TEST PAPER III

Pages 271-272.

1. (*a*) ϕ, $\left\{1, 2, 3, 4, \frac{1}{2}\right\}$, $\left\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, 2, 3, 4\right\}$, ϕ.

(*b*) For example, ·75, 3·4, 11·45, $\frac{3}{11}$, $\frac{57}{40}$ are fractions but not natural numbers. Not possible.

2. {1, 3, 7, 9, 21, 63}, {1, 3, 5, 9, 15, 45}, {1, 3, 9, 27}. {1, 3, 9}, 9, 1. 9.

3. The only factor of 1 is 1.

5. (*a*) Yes. If x, y are fractions, then $x-y$ is meaningful only if $x > y$.

6. (*a*) {(1, 1, 3), (2, 1, 2), (3, 1, 1)}. 7. (*b*) (*iii*).

8. (*a*) $x(2x+3y)/y$ (*b*) $\{x : -2 \leqslant x \leqslant 5/4\}$.

10. (*a*) 3 km. p.h., 8 km. p.h. (*b*) 5, 9.

TEST PAPER IV

Pages 273-274.

1. (*a*) $C \subset (A \cup B)$ is true

(*b*) For example, 0, $-\frac{4}{3}$, -2, $-\frac{7}{11}$, $-3·75$ are rational numbers but not fractions. Not possible

2. {5, 10, 15, 20, ...}, {10, 20, 30, 40, ...}, {15, 30, 45, 60, . .}. {30, 60, 90, ...}. No. 30.

7. (*a*) No (*b*) (*i*) For example, $\frac{2}{3}$, $\frac{6}{11}$, $\frac{5}{8}$, $\frac{4}{7}$, $\frac{3}{5}$

(*b*) (*ii*) 4, 5, 6.

8. (*a*) $\frac{38}{3}$.

(*b*) No The equation has no root if b^2-4ac is not the square of a rational number.

9. (a) $\left\{\left(1, \frac{1}{2}, \frac{1}{3}\right)\right\}$

10. (a) $\left\{x : \frac{1}{2} \leqslant x \leqslant 3\right\}$ (b) Rs. 165000, Rs. 15000.

TEST PAPER V

Pages 274-276.

1. (a) $\left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}$, $\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4\right\}$. $A \subset B$ is true.

   (b) For example, $0, \frac{1}{2}, \frac{3}{4}, -\frac{11}{5}, -\frac{7}{22}, -35, -22{\cdot}3$. No.

3. $\{1, 3, 5, 9, 15, 45\}, \{1, 3, 7, 9, 21, 63\}, \{1, 2, 4, 5, 10, 20\}, \{1\}$. Finite. 1, 1. 1.

4. (b) $3 \times 5^3, 2^4 \times 3 \times 5, 5 \times 2 \times 3 \times 13, 3^2 \times 5 \times 13$. 15

6. (a) $x < \frac{5}{2}$. (b) N. Infinite.

7. (a) No. For example, there is no integer between 0 and 1.

   (b) ϕ.

8. (a) $-\frac{1}{4}$. (b) $\left\{x : -10 \leqslant x \leqslant 3\right\}$.

9. (b) $\left\{\left(-\frac{13}{22}, \frac{13}{7}\right)\right\}$.

10. (a) It empties the cistern in ten hours. (b) 5 hours.